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Higher Conformal Multifractality[1]

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We derive, from conformal invariance and quantum gravity, the multifractal spectrum $f(\alpha)$ of the harmonic measure (i.e., electrostatic potential, or diffusion field) near any conformally invariant fractal in two dimensions. It gives the Hausdorff dimension of the set of points where the potential varies with distance r to the fractal frontier as r^{α}. First examples are a random walk, i.e., a Brownian motion, a self-avoiding walk, or a critical percolation cluster. The generalized dimensions $D(n)$ as well as the multifractal functions $f(\alpha)$ are derived, and are all identical for these three cases. The external frontiers of a Brownian motion and of a percolation cluster are thus *identical to a self-avoiding walk* in the scaling limit. The multifractal (MF) function $f(\alpha, c)$ of the electrostatic potential near any conformally invariant fractal boundary, like a critical $O(N)$ loop or a Q-state Potts cluster, is given as a function of the central charge c of the associated conformal field theory. The dimensions D_{EP} of the external perimeter and D_{H} of the hull of a critical scaling curve or cluster obey the *superuniversal* duality equation $(D_{\mathrm{EP}}-1)(D_{\mathrm{H}}-1)=\frac{1}{4}$. Finally, for a conformally invariant scaling curve which is *simple*, i.e., without double points, we derive higher multifractal functions, like the universal function $f_2(\alpha, \alpha')$ which gives the Hausdorff dimension of the points where the potential varies jointly with distance r as r^{α} on one side of the curve, and as $r^{\alpha'}$ on the other. The general case of the potential distribution between the branches of a star made of an arbitrary number of scaling paths is also treated. The results apply to critical $O(N)$ loops, Potts clusters, and to the SLE_{κ} process. We present a duality between external perimeters of Potts clusters and $O(N)$ loops at their critical point, as well as the corresponding duality in the SLE_{κ} process for $\kappa\kappa' = 16$.

KEY WORDS: Multifractality; conformal invariance; harmonic measure; Brownian motion; self-avoiding walks; percolation; Potts clusters; $O(N)$ model; SLE_{κ} process, hull; external perimeter; duality.

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1. INTRODUCTION

Quantum field theory can be described very generally in terms of the statistics of Brownian paths and of their intersections.[1] This equivalence is used in polymer theory[1] and in rigorous studies of second-order phase transitions and field theories.[2] In probability theory, non trivial properties of Brownian paths have led to intriguing conjectures. Mandelbrot[3] suggested for instance that in two dimensions, the external frontier of a planar Brownian path has a Hausdorff dimension $D = 4/3$, identical to that of a planar self-avoiding walk, i.e., a polymer. Families of universal critical exponents are associated with *intersection* properties of sets of random walks.[4–10] In his famous article "*Walks, walls, wetting and melting*,"[4] Michael Fisher considered groups of 1D "vicious walkers" which had to avoid each other: this amounts to considering 2D *directed* mutually-avoiding random walks. The present paper deals with the scaling properties of 2D mutually-avoiding random walks or fractal sets. Here the latter are conformally invariant, allowing the use of techniques borrowed from "2D quantum gravity," where mutual avoidance is equivalent to solving a linear problem. In ref. 4, the directedness of the walks made the problem tractable. The general similarity in the scaling concepts used, however, certainly allows one to dedicate this work as a tribute to the outstanding influence Michael Fisher had in the general field of random scaling paths.

The concepts of generalized dimensions and associated multifractal measures were developed two decades ago.[11–14] Universal geometrical fractals, e.g., random walks, polymers, Ising or percolation models are essentially related to standard critical phenomena and field theory, for which conformal invariance in two dimensions (2D) has brought a wealth of exact results (see, e.g., refs. 15–21). Multifractals and field theory must have deep connections, since the algebras of their respective correlation functions reveal intriguing similarities.[22]

In classical potential theory, i.e., that of the electrostatic or diffusion field near random fractal boundaries such as diffusion limited aggregates (DLA), or the fractal structures arising in critical phenomena, the self-similarity of the latter is reflected in a *multifractal* (MF) behavior of the potential. In DLA, the potential, also called harmonic measure, actually determines the growth process and its scaling properties are intimately related to those of the of the cluster itself.[23] In statistical fractals, the Laplacian field is created by the random boundary, and should be derivable, in a probabilistic sense, from the knowledge of the latter. A first example was studied in ref. 24, where the fractal boundary, the "absorber," was chosen to be a simple random walk (RW), or a self-avoiding walk (SAW), accessible to renormalization group methods near four dimensions.

In *two dimensions* (2D), conformal field theory (CFT) has lent strong support to the conjecture that statistical systems at their critical point, in their scaling (continuum) limit, produce *conformally invariant* (CI) fractal structures, examples of which are the continuum scaling limits of RW's, SAW's, critical Ising or Potts clusters (which presented a mathematical challenge, see, e.g., refs. 25–27). The harmonic measure near such clusters possesses universal multifractal exponents, as we shall see. In analogy to the beautiful simplicity of the classical method of conformal transforms to solve 2D electrostatics of *Euclidean* domains, a *universal* solution is indeed possible for the planar potential near a CI fractal.

A first exact example was solved within the whole universality class of random or self-avoiding walks, and percolation clusters, which all possess the same harmonic MF spectrum in two dimensions in refs. 28–30. (Related results can be found in refs. 31–33.) After a detailed description of this class and its link to quantum gravity, we address the general solution for the potential distribution near any conformal fractal in 2D.[34] The exact multifractal spectra describing the singularities of the potential, or, equivalently, the distribution of local "electrostatic" wedge angles along the fractal boundary, are given, and shown to depend only on the so-called *central charge* c, a parameter which labels the universality class of the underlying CFT. A further result, first obtained in ref. 34, is the existence of a *duality* between the external frontiers of random clusters and their hulls, which applies in particular to Fortuin-Kasteleyn clusters in the Potts model, and to the so-called stochastic Löwner evolution (*SLE*) process (see below).

A new feature will be the consideration of *higher multifractality*, which occurs in a natural way in the joint distribution of potential on both sides of a random CI scaling path, or more generally, in the distribution of potential between the branches of a *star* made of an arbitrary number of CI paths. The associated universal multifractal spectrum will depend on two variables, or more generally, on m variables in the case of an m-arm star. We shall derive it first for Brownian motion or self-avoiding walks,[35] before addressing the general case.

Consider a two-dimensional very large "absorber" $\mathscr{S}$, which can be a random walk, a self-avoiding walk, a percolation cluster, or, more generally, a (critical) scaling path or cluster. Define the harmonic measure $\mathrm{H}(w)$ as the probability that a random walker launched from infinity, *first* hits the outer "hull's frontier" or (accessible) frontier $\mathscr{F}(\mathscr{S})$ at point $w \in \mathscr{F}(\mathscr{S})$. One then considers a covering of $\mathscr{F}$ by balls $\mathscr{B}(w, a)$ of radius a, and centered at points w forming a discrete subset $\mathscr{F}/\{a\}$ of $\mathscr{F}$. Let $\mathrm{H}(\mathscr{F} \cap \mathscr{B}(w, a))$ be the harmonic measure of the points of $\mathscr{F}$ in the ball $\mathscr{B}(w, a)$. We are interested in the moments of H, averaged over all realizations of RW's and $\mathscr{S}$

$$\mathscr{Z}_n = \left\langle \sum_{w \in \mathscr{F}/\{a\}} \mathrm{H}^n(\mathscr{F} \cap \mathscr{B}(w, a)) \right\rangle, \tag{1}$$

where n can be, *a priori*, a real number. For very large absorbers $\mathscr{S}$ and hull's frontiers $\mathscr{F}(\mathscr{S})$ of average size R, one expects these moments to scale as

$$\mathscr{Z}_n \approx (a/R)^{\tau(n)}, \tag{2}$$

where the radius a serves as a microscopic cut-off, reminiscent of the lattice structure, and where the multifractal scaling exponents $\tau(n)$ encode *generalized dimensions*

$$D(n) = \frac{\tau(n)}{n-1}, \tag{3}$$

which vary in a non-linear way with n.[11–14] Several *a priori* results are known. $D(0)$ is the Hausdorff dimension of the accessible frontier of the fractal. By construction, H is a normalized probability measure, so that $\tau(1) = 0$. Makarov's theorem,[36] here applied to the Hölder regular curve describing the frontier,[37] gives the *non trivial* information dimension $\tau'(1) = D(1) = 1$. The multifractal formalism[11–14] further involves characterizing subsets $\mathscr{F}_\alpha$ of sites of the hull's frontier $\mathscr{F}$ by a Hölder exponent α, such that the H-measure of the frontier points in the ball $\mathscr{B}(w, a)$ of radius a centered at $w_\alpha \in \mathscr{F}_\alpha$ scales as

$$\mathrm{H}(\mathscr{F} \cap \mathscr{B}(w_\alpha, a)) \approx (a/R)^\alpha. \tag{4}$$

The Hausdorff or "fractal dimension" $f(\alpha)$ of the set $\mathscr{F}_\alpha$, such that

$$\mathrm{Card}\, \mathscr{F}_\alpha \approx R^{f(\alpha)}, \tag{5}$$

is given by the symmetric Legendre transform of $\tau(n)$:

$$\alpha = \frac{d\tau}{dn}(n), \qquad \tau(n) + f(\alpha) = \alpha n, \qquad n = \frac{df}{d\alpha}(\alpha). \tag{6}$$

Because of the statistical ensemble average (1), values of $f(\alpha)$ can become negative for some domains of α.[24] As we shall see, the associated exponents $\tau(n)$ above can be recast as those of star copolymers made of independent RW's in a bunch, diffusing away from a generic point of the absorber.

One can equivalently consider potential theory near the same fractal boundary, now charged. One assumes the absorber to be perfectly conducting, and introduces the harmonic potential $H(z)$ at points z in the domain exterior to $\mathscr{F}$, with (Dirichlet) boundary condition $H=0$ on $\mathscr{F}$, and $H=1$ on a large exterior circle. Then the local behavior of the potential

$$H(z \to w_\alpha) \sim r^\alpha, \qquad r=|z-w_\alpha|, \tag{7}$$

depends on the same α-exponent as the harmonic measure around point $w_\alpha \in \mathscr{F}_\alpha$, and $f(\alpha)=\dim \mathscr{F}_\alpha$ appears as the Hausdorff dimension of boundary points inducing the local behavior (7).

Generalizations to higher conformal multifractality can be defined as follows. When it is *simple*, i.e., without double points, the conformally scaling curve $\mathscr{F}$ can be reached from both sides, with a distribution of potential H_+ on one side, and H_- on the other, such that

$$H_+(z \to w_{\alpha,\alpha'}) \sim r^\alpha, \qquad H_-(z \to w_{\alpha,\alpha'}) \sim r^{\alpha'}, \tag{8}$$

when approaching a point $w_{\alpha,\alpha'}$ of subset $\mathscr{F}_{\alpha,\alpha'}$ at distance $r=|z-w_{\alpha,\alpha'}|$ (Fig. 1). Then a double-multifractal spectrum $f_2(\alpha,\alpha')=\dim \mathscr{F}_{\alpha,\alpha'}$ yields the Hausdorff dimension of the set of points of type (α,α'). This can be generalized to the multiple scaling behavior r^{α_i}, $i=1,..., m$ of the potential in the m sectors of an m-arm scaling star, with a multifractal spectrum $f_m(\{\alpha_i\})$, to be calculated below.

We shall use conformal tools (linked to quantum gravity), which allow the mathematical description of random walks interacting with CI fractal

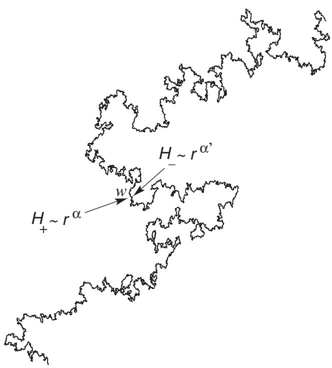

Fig. 1. Double distribution of harmonic potential H on both sides of a simple scaling curve (here a SAW, courtesy of Tom G. Kennedy). The local exponents on both sides of point $w=w_{\alpha,\alpha'}$ are α and α'. The Hausdorff dimension of such points along the SAW is $f_2(\alpha,\alpha')$.

structures, thereby yielding a complete, albeit probabilistic, description of the potential. The results are applied directly to well-recognized universal fractals, like $O(N)$ loops or Potts clusters. In particular, a subtle geometrical structure is observed in Potts clusters, where the *external perimeter* (EP), which bears the electrostatic charge, differs from the full cluster's hull. Its fractal dimension D_{EP} is obtained exactly, generalizing the case of percolation elucidated in ref. 38. We obtain a duality relation:

$$(D_{\mathrm{EP}}-1)(D_{\mathrm{H}}-1)=\tfrac{1}{4}, \tag{9}$$

where $D_{\mathrm{H}} \geqslant D_{\mathrm{EP}}$ is the hull dimension. Notice that the symmetric point of (9) is $3/2$, which is the maximum dimension of a *simple* (i.e., double-point-free) conformally invariant random curve in the plane. This duality, which actually gives clusters EP's as simple $O(n)$ loops at their critical point, was first obtained in ref. 34. It predicts a corresponding duality in the *SLE* process (see below).

The *quantum gravity* techniques used here are not yet widely known in statistical mechanics, since they originally belonged to string or random matrix theory. These techniques, moreover, are not yet within the realm of rigorous mathematics. Contact will be made with rigorous results recently obtained in probability theory for Brownian motion and conformally invariant scaling curves,[31, 32, 39–41] or percolation,[42–44] which, by completely different techniques (using in particular the so-called "stochastic Löwner evolution"[39]), parallel those of statistical mechanics and quantum gravity. In particular, our duality equation (9) brings in the $\kappa\kappa'=16$ duality, where the $SLE_{\kappa'}$ process is the simple frontier of the SLE_{κ}, for $\kappa \geqslant 4$. This of course hints at deep connections between probability theory and conformal field theory. In particular, the correspondence extensively used here, which exists between scaling laws in the plane, and on a random Riemann surface appears to be fundamental.

2. INTERSECTIONS OF RANDOM WALKS

Let us first define intersection exponents for random walks or Brownian motions, which, while simpler than the multifractal exponents considered above, in fact generate the latter. Consider a number L of independent random walks (or Brownian paths) $B^{(l)}$, $l=1, .., L$ in $\mathbb{Z}^d$ (or $\mathbb{R}^d$), starting at fixed neighboring points, and the probability

$$P_L(t)=P\left\{\bigcup_{l,l'=1}^{L}(B^{(l)}[0,t]\cap B^{(l')}[0,t])=\varnothing\right\}, \tag{10}$$

that the intersection of their paths up to time t is empty.[5,7] At large times and for $d < 4$, one expects this probability to decay as

$$P_L(t) \approx t^{-\zeta_L}, \tag{11}$$

where $\zeta_L(d)$ is a *universal* exponent depending only on L and d. Above the upper critical dimension $d = 4$, RWs almost surely do not intersect. The existence of exponents ζ_L in $d = 2, 3$ and their universality have been proven,[10] and they can be calculated near $d = 4$ by renormalization theory.[7] A generalization was introduced[8] for L walks constrained to stay in a half-plane, and starting at neighboring points on the boundary, with a non-intersection probability $\tilde{P}_L(t)$ of their paths governed by a "surface" critical exponent $\tilde{\zeta}_L$ such that

$$\tilde{P}_L(t) \approx t^{-\tilde{\zeta}_L}. \tag{12}$$

It was conjectured from conformal invariance arguments and numerical simulations that in 2D[8]

$$\zeta_L = h_{0,L}^{(c=0)} = \tfrac{1}{24}(4L^2 - 1), \tag{13}$$

and for the half-plane

$$2\tilde{\zeta}_L = h_{1,2L+2}^{(c=0)} = \tfrac{1}{3}L(1+2L), \tag{14}$$

where $h_{p,q}^{(c)}$ denotes the Kač conformal weight

$$h_{p,q}^{(c)} = \frac{[(m+1)\,p - mq]^2 - 1}{4m(m+1)}, \tag{15}$$

of a minimal conformal field theory of central charge $c = 1 - 6/m(m+1)$, $m \in \mathbb{N}^*$.[16] For Brownian motions $c = 0$, and $m = 2$. For $L = 1$, the intriguing $\zeta_1 = 1/8$ is actually the disconnection exponent governing the probability that the origin of a single walk remains accessible from infinity without crossing the walk.

To derive the conjectured intersection exponents above, the idea[28] is to map the original random walk problem in the plane onto a random lattice with planar geometry, or, in other words, in presence of two-dimensional *quantum gravity*.[45] The key point is that the random walk intersection exponents on the random lattice are related to those in the plane. Furthermore, the RW intersection problem can be solved in quantum gravity. Thus, the exponents ζ_L (Eq. (13)) and $\tilde{\zeta}_L$ (Eq. (14) in the standard

Euclidean plane are derived from this mapping to a random lattice or Riemann surface.

Random surfaces, in relation to string theory,[46] have been the subject and source of important developments in statistical mechanics in two-dimensions. In particular, the discretization of string models led to the consideration of abstract random lattices G, the connectivity fluctuations of which represent those of the metric, i.e., pure 2D quantum gravity.[47] One can then put any 2D statistical model (like Ising model,[48] self-avoiding walks[49]) on the random planar graph G, thereby obtaining a new critical behavior, corresponding to the confluence of the criticality of the random surface G with the critical point of the original model. The critical system "dressed by gravity" has a larger conformal symmetry which allowed Knizhnik, Polyakov, and Zamolodchikov (KPZ)[45, 50] to establish the existence of a *relation* between the conformal dimensions $\triangle^{(0)}$ of scaling operators in the plane and those in presence of gravity, $\triangle$:

$$\triangle^{(0)} = \triangle(\triangle - \gamma)/(1-\gamma), \tag{16}$$

where γ is a parameter related to the central charge of the statistical model in the plane:

$$c = 1 - 6\gamma^2/(1-\gamma); \tag{17}$$

for a minimal model of the series (15), $\gamma = -1/m$, and $\triangle^{(0)}_{p,q} \equiv h^{(c)}_{p,q}$.

Let us now consider as a statistical model *random walks* on a *random graph*. We know[8] that the central charge $c = 0$, whence $m = 2$, $\gamma = -1/2$. Thus the KPZ relation becomes

$$\triangle^{(0)} = U(\Delta) \equiv \tfrac{1}{3}\triangle(1+2\triangle), \tag{18}$$

which has exactly the same analytical form as the conjecture (14)! Thus, from the KPZ equation one infers that the planar Brownian intersection exponents Eqs. (13) and (14) are equivalent to Brownian intersection exponents in quantum gravity:

$$\triangle_L = \tfrac{1}{2}(L - \tfrac{1}{2}), \tag{19}$$

$$\tilde{\triangle}_L = L. \tag{20}$$

Let us now sketch the derivation of the latter quantum gravity exponents.[28]

Consider the set of planar random graphs G, built up with, e.g., trivalent vertices tied together in a *random way*. The topology is fixed here to be that of a sphere ($\mathscr{S}$) or a disc ($\mathscr{D}$). The partition function is defined as

$$Z_\chi(\beta) = \sum_G \frac{1}{S(G)} e^{-\beta\,|G|}, \tag{21}$$

where χ denotes the Euler characteristic $\chi = 2\ (\mathscr{S}), 1\ (\mathscr{D})$; $|G|$ is the number of vertices of G, $S(G)$ its symmetry factor. The partition sum converges for all values of the parameter β larger than some critical β_c. At $\beta \to \beta_c^+$, a singularity appears due to the presence of infinite graphs in (21)

$$Z_\chi(\beta) \sim (\beta - \beta_c)^{2-\gamma_{\rm str}(\chi)}, \tag{22}$$

where $\gamma_{\rm str}(\chi)$ is the string susceptibility exponent. For pure gravity as described in (21), the embedding dimension $d = 0$ coincides with the central charge $c = 0$, and $\gamma_{\rm str}(\chi) = 2 - \frac{5}{4}\chi$.[51]

Now, put a set of L random walks $\mathscr{B} = \{B_{ij}^{(l)}, l = 1,..., L\}$ on the *random graph* G with the special constraint that they start at the same vertex $i \in G$, end at the same vertex $j \in G$, and have no intersections in between. We introduce the L-walk partition function on the random lattice[28]:

$$Z_L(\beta, z) = \sum_{\text{planar } G} \frac{1}{S(G)} e^{-\beta\,|G|} \sum_{i,\, j \in G} \sum_{\substack{B_{ij}^{(l)} \\ l = 1,..., L}} z^{|\mathscr{B}|}, \tag{23}$$

where a fugacity z is associated with the total number $|\mathscr{B}| = |\bigcup_{l=1}^{L} B^{(l)}|$ of vertices visited by the walks.

We generalize this to the *boundary* case where G now has the topology of a disc and where the random walks connect two sites i and j now on the boundary ∂G:

$$\tilde{Z}_L(\beta, \beta', z) = \sum_{\text{disc } G} e^{-\beta\,|G|} e^{-\beta'\,|\partial G|} \sum_{i,\, j \in \partial G} \sum_{\substack{B_{ij}^{(l)} \\ l = 1,..., L}} z^{|\mathscr{B}|}, \tag{24}$$

where $e^{-\beta'}$ is the fugacity associated with the boundary's length.

The double grand canonical partition function (23) associated with non-intersecting RW's on a random lattice can be calculated exactly.[28] The critical behavior of $Z_L(\beta, z)$ is then obtained by taking the double scaling limit $\beta \to \beta_c$ (infinite random surface) and $z \to z_c$ (infinite RW's). The analysis of this singular behavior in terms of conformal dimensions is performed by using *finite size scaling* (FSS),[49] where one must have $|\mathscr{B}| \sim |G|^{\frac{1}{2}}$. One obtains[28]:

$$Z_L(\beta, z) \sim (\beta - \beta_c)^L \sim |G|^{-L}. \tag{25}$$

Z_L (23) represents a random surface with two *punctures* where two conformal operators of dimension $\triangle_L$ are located (here two vertices of L non-intersecting RW's), and in a graphical way scales as

$$Z_L \sim Z[\;] \times |G|^{-2\triangle_L} \tag{26}$$

where the partition function of the two-puncture surface is the second derivative of $Z_{\chi=2}(\beta)$ (22). The latter two equations yield

$$2\triangle_L - \gamma_{\mathrm{str}}(\chi = 2) = L, \tag{27}$$

where $\gamma_{\mathrm{str}}(\chi = 2) = -1/2$. We thus get the previously announced result

$$\triangle_L = \tfrac{1}{2}(L - \tfrac{1}{2}). \tag{28}$$

For the boundary partition function $\tilde{Z}_L$ (24) a similar analysis can be performed near the triple critical point where the boundary length also diverges. The boundary partition function $\tilde{Z}_L$ corresponds to two boundary operators of dimensions $\tilde{\triangle}_L$, integrated over ∂G, on a random surface with the topology of a disc, or in graphical terms:

$$\tilde{Z}_L \sim Z(\;) \times |\partial G|^{-2\tilde{\triangle}_L}. \tag{29}$$

From the exact calculation of the *boundary* partition function (24), one gets the further equivalence to the *bulk* one:

$$\tilde{Z}_L / Z(\;) \sim Z_L, \tag{30}$$

where the equivalences hold true in terms of scaling behavior. Comparing Eqs. (29), (30), and (25), and using the FSS $|\partial G| \sim |G|^{1/2}$ gives

$$\tilde{\triangle}_L = L. \tag{31}$$

Applying the quadratic KPZ relation (18) to $\triangle_L$ and $\tilde{\triangle}_L$ above yields at once the values in the plane $\mathbb{R}^2$, $\triangle_L^{(0)} \equiv \zeta_L$ (Eq. (13)), and $\tilde{\triangle}_L^{(0)} \equiv 2\tilde{\zeta}_L$ (Eq. (14)).

Consider now the exponents

$$\zeta(n_1, .., n_L) = \triangle^{(0)}\{n_l\},$$

as well as

$$2\tilde{\zeta}(n_1, .., n_L) = \tilde{\Delta}^{(0)}\{n_l\},$$

describing L mutually-avoiding bunches $l = 1, .., L$, each made of n_l *independent* walks, i.e., mutually "transparent,"[52] with possible mutual intersections in a bunch. In presence of gravity each bunch will contribute its own *normalized boundary partition function* as a factor, and yield a natural generalization of (30)

$$Z\{n_l\} \sim \frac{\tilde{Z}\{n_l\}}{Z(\bigcirc)} \sim \prod_{l=1}^{L} \frac{\tilde{Z}(n_l)}{Z(\bigcirc)}, \tag{32}$$

to be identified with $|\partial G|^{-2\tilde{\Delta}\{n_l\}}$. The *factorization* property (32) immediately implies the *additivity of boundary conformal dimensions in presence of gravity*

$$\tilde{\Delta}\{n_1, .., n_L\} = \sum_{l=1}^{L} \tilde{\Delta}(n_l), \tag{33}$$

where $\tilde{\Delta}(n)$ is now the boundary dimension of a *single* bunch of n transparent walks on the random surface. We know $\tilde{\Delta}(n)$ exactly since it corresponds in the standard plane to a trivial surface conformal dimension $\tilde{\Delta}^{(0)}(n) = n$. It thus suffices to *invert* (18) to get

$$\tilde{\Delta}(n) = U^{-1}(n) = \tfrac{1}{4}(\sqrt{24n+1}-1). \tag{34}$$

One notes the identification (32), on a random surface, of the bulk partition function with the ratio of boundary ones, which gives the generalization of (27): $\tilde{\Delta}\{n_1,..., n_L\} = 2\Delta\{n_1,..., n_L\} - \gamma_{\mathrm{str}}\ (x = 2)$. In the plane, using once again the KPZ relation (18) for $\tilde{\Delta}\{n_l\}$ and $\Delta\{n_l\}$ gives the general results[28]

$$\zeta(n_1, .., n_L) = V(x) \equiv \tfrac{1}{24}(4x^2 - 1), \tag{35}$$

$$2\tilde{\zeta}(n_1, .., n_L) = U(x) = \tfrac{1}{3}\, x(1 + 2x), \tag{36}$$

$$x = \sum_{l=1}^{L} U^{-1}(n_l) = \sum_{l=1}^{L} \tfrac{1}{4}(\sqrt{24n_l + 1} - 1). \tag{37}$$

Lawler and Werner[31] proved by probabilistic means, using the geometrical conformal invariance of Brownian motions, the *existence* of two functions U and V satisfying the structure (35)–(37). The quantum gravity approach here explains this structure in terms of linear equation (33), and yields the explicit functions $U(x)$ and $V(x) \equiv U(\tfrac{1}{2}(x - \tfrac{1}{2}))$ of (35) and (36). The same expression of these functions has been finally derived in probability theory.[40]

Let us remark that the above equations yield for $\zeta(2, 1^{(L)})$ describing a two-sided walk and L one-sided walks, all mutually non-intersecting,

$$\zeta(2, 1^{(L)}) = \zeta_{L+\frac{3}{2}} = V(L+\tfrac{3}{2}) = \tfrac{1}{6}(L+1)(L+2). \tag{38}$$

For $L=1$, $\zeta(2,1) = \zeta_{5/2} = 1$ gives correctly the escape probability of a RW from another RW. For $L=0$, $\zeta(2, 1^{(0)}) = \zeta_{3/2} = 1/3$ is related to the Hausdorff dimension of the frontier by $D = 2-2\zeta$.[53] Thus we obtain[28]

$$D = 2 - 2\zeta_{\frac{3}{2}} = \tfrac{4}{3}, \tag{39}$$

i.e., *Mandelbrot's conjecture*. This conjecture has finally been proven in probability theory,[41] using the analytic properties of the exponents derived from the so-called stochastic Löwner equation.[39] The quantum geometric structure explicited here allows several generalizations, which we now describe.[29]

3. RANDOM WALKS AND SELF-AVOIDING WALKS

We now generalize the scaling structure obtained in the preceding section to arbitrary sets of random or self-avoiding walks[29] (see also refs. 31 and 32). Consider a general star copolymer $\mathscr{S}$ in the plane $\mathbb{R}^2$ (or in $\mathbb{Z}^2$), made of an arbitrary mixture of Brownian paths or RW's (set $\mathscr{B}$), and polymers or SAW's (set $\mathscr{P}$), all starting at neighboring points. Any pair (A, B) of such paths, $A, B \in \mathscr{B}$ or $\mathscr{P}$, can be constrained in a specific way: either they avoid each other ($A \cap B = \emptyset$, noted $A \wedge B$), or they are independent, i.e., "transparent" and can cross each other (noted $A \vee B$).[29, 54] This notation allows any *nested* interaction structure;[29] one can decide for instance that the branches $\{P_\ell \in \mathscr{P}\}_{\ell=1,\dots,L}$ of an L-star polymer, all mutually-avoiding, further avoid a bunch of Brownian paths $\{B_k \in \mathscr{B}\}_{k=1,\dots,n}$, all transparent to each other:

$$\mathscr{S} = \left(\bigwedge_{\ell=1}^{L} P_\ell \right) \wedge \left(\bigvee_{k=1}^{n} B_k \right). \tag{40}$$

In 2D the order of the branches of the star copolymer *does* matter and is intrinsic to our $(\wedge, \vee)$ notation.

To each *specific* star copolymer center $\mathscr{S}$ is attached a conformal scaling operator with a scaling dimension $x(\mathscr{S})$. To obtain proper scaling we consider the partition functions of Brownian paths and polymers having the same mean size R. When the star is constrained to stay in a *half-plane* with its core placed near the *boundary*, its partition function will scale with new boundary scaling dimension $\tilde{x}(\mathcal{S})$.[8,19,20]

Any scaling dimension x in the bulk is twice the *conformal dimension* (c.d.) $\Delta^{(0)}$ of the corresponding operator, while near a boundary (b.c.d.) they are identical:

$$x = 2\Delta^{(0)}, \qquad \tilde{x} = \tilde{\Delta}^{(0)}. \tag{41}$$

As above, the idea is to use the representation where the RW's or SAW's are on a 2D random lattice, or a random Riemann surface, i.e., in the presence of 2D *quantum gravity*.[45] The general relation (18) depends only on the central charge, and is valid for polymers, for which $c=0$. Let us summarize the results,[29] expressed here in terms of the scaling dimensions in the standard plane. For a critical system with central charge $c=0$, the two universal functions:

$$U(x) = \tfrac{1}{3}\, x(1+2x), \qquad V(x) = \tfrac{1}{24}\,(4x^2-1), \tag{42}$$

with $V(x) \equiv U(\frac{1}{2}\,(x-\frac{1}{2}))$, generate all the scaling exponents. The scaling exponents $x(A \wedge B)$, and $\tilde{x}(A \wedge B)$, of two *mutually avoiding* stars A, B, with proper scaling exponents $x(A)$, $x(B)$, or boundary exponents $\tilde{x}(A)$, $\tilde{x}(B)$, obey the *star algebra*[28,29]

$$\begin{aligned} x(A \wedge B) &= 2V[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))] \\ \tilde{x}(A \wedge B) &= U[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))], \end{aligned} \tag{43}$$

where $U^{-1}(x)$ is the inverse function of U

$$U^{-1}(x) = \tfrac{1}{4}(\sqrt{24x+1}-1). \tag{44}$$

On a random surface, $U^{-1}(\tilde{x})$ is the boundary dimension corresponding to the value $\tilde{x}$ in $\mathbb{R} \times \mathbb{R}^+$, and the sum of U^{-1} functions in Eq. (43) represents linearly the juxtaposition $A \wedge B$ of two sets of random paths near their random frontier, i.e., the product of two "boundary operators" on the random surface. The latter sum is mapped by the functions U, V, into the scaling dimensions in $\mathbb{R}^2$.[29]

The rules (43), which mix bulk and boundary exponents, come from simple factorization properties on a random Riemann surface, i.e., in quantum gravity,[28,29] (and are also recurrence relations in $\mathbb{R}^2$ between conformal Riemann maps of the successive mutually-avoiding paths onto the line $\mathbb{R}$ (ref. 31)).

If, on the contrary, A and B are *independent* and can overlap, then by trivial factorization of probabilities their dimensions are additive[29]

$$\begin{aligned} x(A \vee B) &= x(A) + x(B), \\ \tilde{x}(A \vee B) &= \tilde{x}(A) + \tilde{x}(B). \end{aligned} \tag{45}$$

It is clear at this stage that the set of equations above is *complete*. It allows for the calculation of any conformal dimensions associated with a star structure $\mathscr{S}$ of the most general type, as in (40), involving $(\wedge, \vee)$ operations separated by nested parentheses.[29]

Brownian-polymer exponents: The single extremity scaling dimensions are for a RW or a SAW near a Dirichlet boundary in $\mathbb{R}^2$[20, 55]

$$\tilde{x}_B(1) = \tilde{\Delta}_B^{(0)}(1) = 1, \qquad \tilde{x}_P(1) = \tilde{\Delta}_P^{(0)}(1) = \tfrac{5}{8}, \tag{46}$$

or on G, $\tilde{\Delta}_B(1) = U^{-1}(1) = 1$, $\tilde{\Delta}_P(1) = U^{-1}(\frac{5}{8}) = \frac{3}{4}$. Because of the star algebra described above these are the only numerical seeds, i.e., generators, we need.

Stars can include bunches of n copies of transparent RW's or m transparent SAW's. Their b.c.d.'s in $\mathbb{R}^2$ are respectively, by using (45) and (46), $\tilde{\Delta}_B^{(0)}(n) = n$ and $\tilde{\Delta}_P^{(0)}(m) = \frac{5}{8} m$, from which the inverse mapping to the random surface yields $\tilde{\Delta}_B(n) = U^{-1}(n)$ and $\tilde{\Delta}_P(m) = U^{-1}(\frac{5}{8} m)$. The star made of L bunches $\ell \in \{1,..., L\}$, each of them made of n_ℓ transparent RW's and of m_ℓ transparent SAW's, and the L bunches being mutually-avoiding, has planar scaling dimensions

$$\begin{aligned} \tilde{\Delta}^{(0)}\{n_\ell, m_\ell\} &= U(\tilde{\Delta}), \qquad \Delta^{(0)}\{n_\ell, m_\ell\} = V(\tilde{\Delta}), \\ \tilde{\Delta}\{n_\ell, m_\ell\} &= \sum_{\ell=1}^{L} U^{-1}(n_\ell + \tfrac{5}{8} m_\ell). \end{aligned}$$

This encompasses all previously known exponents for RW's and SAW's.[8, 19, 20] We in particular arrive at the striking *scaling equivalence: a self-avoiding walk is exactly equivalent to* $5/8$ *of a Brownian motion.* Similar results have been obtained in probability theory, based on the general structure of "completely conformally invariant processes," which correspond exactly to $c = 0$ central charge conformal field theories.[32, 40] The construction of the scaling limit of SAW still eludes a rigorous approach, eventhough it is predicted that it corresponds to the "stochastic Löwner evolution" SLE_κ with $\kappa = 8/3$, equivalent to a Coulomb gas with $g = 4/\kappa = 3/2$ (see Section 11 below).

4. CONFORMAL MULTIFRACTALITY AND THE HARMONIC MEASURE

The *harmonic measure*, i.e., the diffusion or electrostatic field near an equipotential fractal boundary,[56] or, equivalently, the electric charge appearing on the frontier of a perfectly conducting fractal, possesses a self-similarity property, which is reflected in a *multifractal* behavior. Cates and Witten[24] considered the case of the Laplacian diffusion field near a simple random walk, or near a self-avoiding walk. The associated exponents can be recast as those of star copolymers made of a bunch of independent RW's diffusing away from a generic point of the absorber. The exact solution to this problem in two dimensions is as follows.[29]

From a mathematical point of view, it can also be derived from the results of refs. 31, 32, 40, and 41 taken altogether.

The two-dimensional "absorber" $\mathscr{S}$ can be a random walk, or a self-avoiding walk. The harmonic measure $\mathrm{H}(w)$ is the probability that another random walker launched from infinity, *first* hits the outer "hull's frontier" or (accessible) frontier $\mathscr{F}(\mathscr{S})$ at point $w \in \mathscr{F}(\mathscr{S})$. A covering of $\mathscr{F}$ by balls $\mathscr{B}(w, a)$ of radius a is centered at points $w \in \mathscr{F}/\{a\}$ forming a discrete subset $\mathscr{F}/\{a\}$ of $\mathscr{F}$. Let $\mathrm{H}(\mathscr{F} \cap \mathscr{B}(w, a))$ be the harmonic measure of the intersection set between $\mathscr{F}$ and the ball $\mathscr{B}(w, a)$. The moments of H, averaged over all realizations of RW's and $\mathscr{S}$ are defined by

$$\mathscr{Z}_n = \left\langle \sum_{w \in \mathscr{F}/\{a\}} \mathrm{H}^n(\mathscr{F} \cap \mathscr{B}(w, a)) \right\rangle, \tag{47}$$

where n can be, *a priori*, a real number. In the limit of large absorbers $\mathscr{S}$ and frontiers $\mathscr{F}(\mathscr{S})$ of average size R, or small covering radius a, i.e, $a/R \to 0$, these moments scale as

$$\mathscr{Z}_n \approx (a/R)^{\tau(n)}, \tag{48}$$

where the multifractal scaling exponents $\tau(n)$ encode generalized dimensions $D(n)$, $\tau(n) = (n-1)\, D(n)$, which vary in a non-linear way with n.[11–14]

As explained in the introduction, the harmonic multifractal spectrum $f(\alpha)$ (Eqs. (4)–(6)) is derived as a Legendre transform of the $\tau(n)$ function. (The existence of the harmonic multifractal spectrum $f(\alpha)$ for a Brownian path has been rigorously established in ref. 57.)

By the very definition of the H-measure, n independent RW's diffusing away from the absorber give a geometric representation of the nth moment H^n, for n *integer*, and convexity arguments give the whole continuation to real values. When the absorber is a RW or a SAW of size R, the site average of its moments H^n is represented by a copolymer star partition function $\mathcal{Z}_R(\mathcal{S}_\wedge n)$, where we have introduced the short-hand notation $\mathcal{S}_\wedge n \equiv \mathcal{S} \wedge (\vee B)^n$ describing the copolymer star made by the absorber $\mathcal{S}$ hit by the bunch $(\vee B)^n$ at the apex only.[24, 29] More precisely one has

$$\mathcal{Z}_n \approx R^2 \mathcal{Z}_R(\mathcal{S}_\wedge n) \tag{49}$$

where the absorber $\mathcal{S}$ is either the two-RW star $B \vee B$ or the two-SAW star $P \wedge P$, made of two non-intersecting SAW's. Owing to Eq. (48), we get the scaling relation

$$\tau(n) = x(\mathcal{S}_\wedge n) - 2. \tag{50}$$

Our formalism (43) immediately gives the scaling dimensions

$$x(\mathcal{S}_\wedge n) = 2V(\tilde{\Delta}(\mathcal{S}) + U^{-1}(n)), \tag{51}$$

where $\tilde{\Delta}(\mathcal{S})$ is as above the quantum gravity boundary dimension of the absorber $\mathcal{S}$ alone. For a RW absorber, we have $\tilde{\Delta}(B \vee B) = U^{-1}(2) = \frac{3}{2}$, while for a SAW $\tilde{\Delta}(P \wedge P) = 2\tilde{\Delta}_P(1) = 2U^{-1}(\frac{5}{8}) = \frac{3}{2}$. The coincidence of these two values tells us that *in* 2D *the harmonic multifractal spectra* $f(\alpha)$ *of a random walk or a self-avoiding walk are identical.* The calculation gives[29]

$$\tau(n) = \frac{1}{2}(n-1) + \frac{5}{24}(\sqrt{24n+1} - 5), \tag{52}$$

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{5}{2}\frac{1}{\sqrt{24n+1}}, \tag{53}$$

$$D(n) = \frac{1}{2} + \frac{5}{\sqrt{24n+1}+5}, \qquad n \in \left[-\frac{1}{24}, +\infty\right), \tag{54}$$

$$f(\alpha) = \frac{25}{48}\left(3 - \frac{1}{2\alpha - 1}\right) - \frac{\alpha}{24}, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right). \tag{55}$$

The corresponding universal curves are shown in Figs. 2 and 3: $\tau(n)$ is half a parabola, and $f(\alpha)$ a hyperbola. $D(1) = \tau'(1) = 1$ is Makarov's theorem. The singularity at $\alpha = \frac{1}{2}$ in the multifractal functions $f(\alpha)$ corresponds to points on the fractal boundary $\mathcal{F}$ where the latter has the local geometry of a needle. The mathematical version of this statement is given by Beurling's theorem,[58] which states that at distance ϵ from the boundary, the harmonic measure is bounded above by

$$H(z\colon \inf_{w \in \mathcal{F}} |z - w| \leqslant \epsilon)) \leqslant C\epsilon^{1/2}, \tag{56}$$

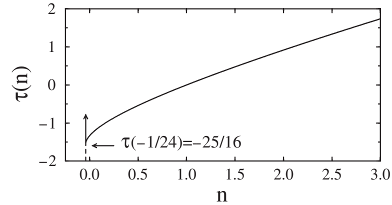

Fig. 2. Harmonic multifractal dimensions $\tau(n)$ of a two-dimensional RW or SAW.

where C is a constant. This insures that the spectrum of multifractal Hölder exponents α is bounded below by $\frac{1}{2}$. The right branch of the $f(\alpha)$ curve has a linear asymptote

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} f(\alpha) = -\frac{1}{24}. \tag{57}$$

Its linear shape is quite reminiscent of that of the multifractal function of the growth probability as in the case of a 2D DLA cluster.[59] The domain of large values of α corresponds to the lowest part $n \to n^* = -\frac{1}{24}$ of the spectrum of dimensions, which is dominated by almost inaccessible sites, and the existence of a linear asymptote to the multifractal function f implies a peculiar behavior for the number of those sites in a lattice setting. Indeed define $\mathcal{N}(H)$ as the number of sites having a probability H to be hit:

$$\mathcal{N}(H) = \mathrm{Card}\{w \in \mathcal{F} : \mathrm{H}(w) = H\}. \tag{58}$$

Using the MF formalism to change from the variable H to α (at fixed value of a/R), shows that $\mathcal{N}(H)$ obeys, for $H \to 0$, a power law behavior

$$\mathcal{N}(H)|_{H \to 0} \approx H^{-\tau^*} \tag{59}$$

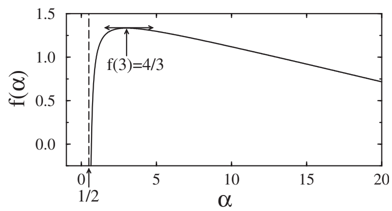

Fig. 3. Harmonic multifractal spectrum $f(\alpha)$ of a two-dimensional RW or SAW.

with an exponent

$$\tau^* = 1 + \lim_{\alpha \to +\infty} \frac{1}{\alpha} f(\alpha) = 1 + n^*. \tag{60}$$

Thus we predict

$$\tau^* = \tfrac{23}{24}. \tag{61}$$

One remarks that $-\tau(0) = \sup_\alpha f(\alpha) = f(3) = \frac{4}{3}$ is the Hausdorff dimension of the *Brownian frontier* or of a SAW. Thus Mandelbrot's classical conjecture identifying the latter two is derived and generalized to the *whole* $f(\alpha)$ harmonic spectrum.

An Invariance Property of $f(\alpha)$

The expression of $f(\alpha)$ simplifies if one considers the combination:

$$f(\alpha) - \alpha = \frac{25}{24}\left[1 - \frac{1}{2}\left(2\alpha - 1 + \frac{1}{2\alpha - 1}\right)\right]. \tag{62}$$

Thus the multifractal function possesses the invariance symmetry[60]

$$f(\alpha) - \alpha = f(\alpha') - \alpha', \tag{63}$$

for α and α' satisfying the duality relation:

$$(2\alpha - 1)(2\alpha' - 1) = 1, \tag{64}$$

or, equivalently

$$\alpha^{-1} + \alpha'^{-1} = 2. \tag{65}$$

When associating a wedge angle $\theta = \pi/\alpha$ to each local singularity exponent α, one recovers the complementary rule for angles in the plane[60]

$$\theta + \theta' = \frac{\pi}{\alpha} + \frac{\pi}{\alpha'} = 2\pi. \tag{66}$$

It is interesting to note that, owing to the explicit forms (53) of α and (54) of $D(n)$, the condition (65) becomes after a little algebra,

$$D(n) + D(n') = 2. \tag{67}$$

This basic symmetry (63) reflects that of the cluster boundary itself under the *exchange of interior and exterior domains* (ref. 60).

Higher Multifractality of Brownian Motion and Self-Avoiding Walk

It is interesting to note that one can also define *higher multifractal* spectra as those depending on several α variables.[35] A first example is given by the double moments of the harmonic measure on *both* sides of a random fractal, taken here as a Brownian motion or a self-avoiding walk. (The general case will be further described in Section 6.) The fractal boundary has to be reached from both sides, so it must be a *simple* curve without double points, which is naturally the case of a SAW. For a Brownian motion, one can consider the subset of the *pinching* or *cut points*, of Hausdorff dimension $D=2-2\zeta_2=3/4$, where the path splits into two non-intersecting parts. Locally the Brownian path then is accessible from both directions.

Let us define:

$$\mathcal{Z}_{n,n'} = \left\langle \sum_{w \in \mathcal{F}/\{a\}} [\mathrm{H}_+(w)]^n \, [\mathrm{H}_-(w)]^{n'} \right\rangle, \tag{68}$$

where $\mathrm{H}_+(w) \equiv \mathrm{H}_+(\mathcal{F} \cap \mathcal{B}(w,a))$ and $\mathrm{H}_-(w) \equiv \mathrm{H}_-(\mathcal{F} \cap \mathcal{B}(w,a))$ are respectively the harmonic measures on "left" or "right" sides of the random fractal. These moments have a multifractal scaling behavior

$$\mathcal{Z}_n \approx (a/R)^{\tau_2(n,n')}, \tag{69}$$

where the exponents $\tau_2(n,n')$ now depend on two moment orders n and n'. The generalization of the Legendre transform Eq. (6) reads

$$\alpha = \frac{\partial \tau_2}{\partial n}(n,n'), \qquad \alpha' = \frac{\partial \tau_2}{\partial n'}(n,n'),$$

$$f_2(\alpha,\alpha') = \alpha n + \alpha' n' - \tau_2(n,n'), \tag{70}$$

$$n = \frac{\partial f_2}{\partial \alpha}(\alpha,\alpha'), \qquad n' = \frac{\partial f_2}{\partial \alpha'}(\alpha,\alpha').$$

We find the τ exponents from the star algebra (43):

$$\tau_2(n,n') = 2V(a' + U^{-1}(n) + U^{-1}(n')) - 2, \tag{71}$$

where a' corresponds to the quantum gravity boundary scaling dimension of the fractal set, i.e., the simple curve or the pinching point set, where the harmonic measure is evaluated on both sides. For a Brownian motion, pinched into two parts separated by the two sets of auxiliary Brownian motions, representing the moments of the harmonic mesures, we have:

$$a' = \tilde{\Delta}(B \wedge B) = 2 \times \tilde{\Delta}_B(1) = 2U^{-1}(1) = 2. \tag{72}$$

For a self-avoiding walk made of two mutually-avoiding one-sided arms, we have

$$a' = \tilde{\Delta}(P \wedge P) = 2 \times \tilde{\Delta}_P(1) = 2U^{-1}(\tfrac{5}{8}) = \tfrac{3}{2}. \tag{73}$$

After performing the double Legendre transform and some calculations, we find

$$f_2(\alpha, \alpha') = 2 + \frac{1}{12} - \frac{1}{3} a''^2 \left[1 - \frac{1}{2} \left(\frac{1}{\alpha} + \frac{1}{\alpha'} \right) \right]^{-1} - \frac{1}{24} (\alpha + \alpha'), \tag{74}$$

$$\alpha = 2 \frac{1}{\sqrt{24n+1}} \left[a'' + \frac{1}{4} (\sqrt{24n+1} + \sqrt{24n'+1}) \right], \tag{75}$$

and a similar symmetric equation for α'. Here a'' has the shifted value:

$$a'' = a' + \gamma = a' - \tfrac{1}{2} \tag{76}$$

$$= \tfrac{3}{2} \quad \text{(RW)}, \qquad \text{or} \quad a'' = 1 \quad \text{(SAW)}. \tag{77}$$

This doubly multifractal spectrum possesses the requested properties, like $\sup_{\alpha'} f(\alpha, \alpha') = f(\alpha)$, where $f(\alpha)$ is (55) above.

This can be generalized to a *star configuration* made of m random walks or m self-avoiding walks, where one looks at the simultaneous behavior of the potential in each sector between the arms of the star (see Section 6 below for a more precise description in the general case). The *poly-multifractal* results read for Brownian motions or self-avoiding polymers:

$$f_m(\{\alpha_{i=1,\ldots,m}\}) = 2 + \frac{1}{12} - \frac{1}{3} a''^2 \left(1 - \frac{1}{2} \sum_{i=1}^{m} \alpha_i^{-1} \right)^{-1} - \frac{1}{24} \sum_{i=1}^{m} \alpha_i, \tag{78}$$

with

$$\alpha_i = 2 \frac{1}{\sqrt{24n_i+1}} \left(a'' + \frac{1}{4} \sum_{j=1}^{m} \sqrt{24n_j+1} \right), \tag{79}$$

and where

$$a' = m, \qquad a'' = a' - \tfrac{1}{4}m = \tfrac{3}{4}m, \tag{80}$$

for *m random walks in a star configuration*, and

$$a' = \tfrac{3}{4}m, \qquad a'' = a' - \tfrac{1}{4}m = \tfrac{1}{2}m, \tag{81}$$

for *m self-avoiding walks in a star configuration*. The two-sided case (77) above is recovered for $m = 2$. The domain of definition of the poly-multifractal function f is given by

$$1 - \tfrac{1}{2} \sum_{i=1}^{m} \alpha_i^{-1} \geqslant 0, \tag{82}$$

as verified by Eq. (79).

5. PERCOLATION CLUSTERS

Consider now a very large two-dimensional incipient cluster $\mathscr{C}$, at the percolation threshold p_c. Define $\mathrm{H}(w)$ as the probability that a random walker, launched from infinity, *first* hits the outer (accessible) percolation hull's frontier $\mathscr{F}(\mathscr{C})$ at point $w \in \mathscr{F}(\mathscr{C})$. The moments of H are averaged over all realizations of RW's and $\mathscr{C}$, as in Eq. (47) above. For very large clusters $\mathscr{C}$ and frontiers $\mathscr{F}(\mathscr{C})$ of average size R, one expects again these moments to scale as in Eq. (48): $\mathscr{Z}_n \approx (a/R)^{\tau(n)}$. These exponents $\tau(n)$ have been obtained recently,[30] using an exact result on the external boundary of a percolation cluster.[38]

We consider site percolation on the 2D triangular lattice. Figure 4 depicts n independent random walks, in a bunch, *first* hitting the external frontier of a percolation cluster at a site $w = (\bullet)$. In order that this site belongs to the *accessible* part of the hull, it must remain, in the *continuous scaling limit*, the source of at least *three non-intersecting crossing paths*, noted $\mathscr{S}_3$, reaching to a (large) distance R (Fig. 4).[38] (Notice that the definition of the *standard* hull requires only a pair of dual lines). The n independent RW's, or Brownian paths B in the scaling limit, in a bunch denoted $(\vee B)^n$, *avoid* the set $\mathscr{S}_3 \equiv (\wedge \mathscr{P})^3$ of three *non-intersecting* connected paths in the percolation system, and this system is governed by a new critical exponent $x(\mathscr{S}_3 \wedge n)$ depending on n.

In terms of these definitions, the harmonic measure moments simply scale with an exponent[29]

$$\tau(n) = x(\mathscr{S}_3 \wedge n) - 2. \tag{83}$$

For percolation, two values of half-plane crossing exponents $\tilde{x}_\ell$ are known by *elementary* means: $\tilde{x}_2 = 1$, $\tilde{x}_3 = 2$.[26] We fuse the two objects $\mathcal{S}_3$ and $(\vee B)^n$ into a new star $\mathcal{S}_3 \wedge (\vee B)^n$, and use (43) to obtain

$$x(\mathcal{S}_3 \wedge n) = 2V(U^{-1}(\tilde{x}_3) + U^{-1}(n)). \tag{84}$$

Specifying $U^{-1}(\tilde{x}_3) = \frac{3}{2}$ finally gives from (42), (44)

$$x(\mathcal{S}_3 \wedge n) = 2 + \tfrac{1}{2}(n-1) + \tfrac{5}{24}(\sqrt{24n+1} - 5).$$

From this $\tau(n)$ (83) is found to be *identical* to (52) for RW's and SAW's; $D(n)$ is then:

$$D(n) = \frac{1}{2} + \frac{5}{\sqrt{24n+1}+5}, \qquad n \in \left[-\frac{1}{24}, +\infty\right), \tag{85}$$

valid for all values of moment order n, $n \geqslant -\frac{1}{24}$. The Legendre transform reads again exactly as in Eq. (55):

$$f(\alpha) = \frac{25}{48}\left(3 - \frac{1}{2\alpha - 1}\right) - \frac{\alpha}{24}, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right). \tag{86}$$

Only in the case of percolation has the harmonic measure been systematically studied numerically, by Meakin *et al.*[61] We show in Fig. 5 the exact curve $D(n)$ (85)[30] together with the numerical results for $n \in \{2,..., 9\}$,[61] showing fairly good agreement.

The average number $\mathcal{N}(H)$ (59) has been also determined numerically for percolation clusters in ref. 62, and for $c = 0$, our prediction (61) $\tau^* = \frac{23}{24} = 0.95833...$ compares very well with the result $\tau^* = 0.951 \pm 0.030$, obtained for $10^{-5} \leqslant H \leqslant 10^{-4}$.

The dimension of the support of the measure $D(0) = \frac{4}{3} \neq D_{\mathrm{H}}$, where $D_{\mathrm{H}} = \frac{7}{4}$ is the Hausdorff dimension of the standard hull, i.e., the outer boundary of critical percolating clusters.[63] The value $D(0) = \frac{4}{3}$ corresponds to the dimension of the *accessible external perimeter*. A direct derivation of its exact value is given in ref. 38. The complement of the accessible perimeter in the hull is made of deep fjords, which do close in the scaling limit and are not probed by the harmonic measure. This is in agreement with the instability phenomenon observed on a lattice for the hull dimension.[64] A striking fact is the complete identity of the multifractal spectrum for percolation to the corresponding results, Eqs. (52)–(55), *both* for random walks and self-avoiding walks. Seen from outside, these three fractal simple curves are not distinguished by the harmonic measure. In fact they are the

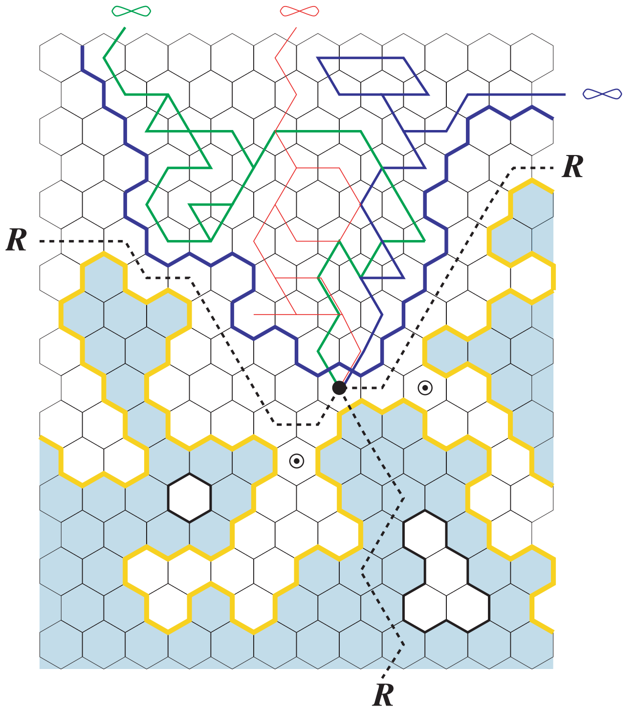

Fig. 4. An accessible site (●) on the external perimeter for site percolation on the triangular lattice. It is defined by the existence, in the *scaling limit*, of three non-intersecting, and connected paths $\mathscr{S}_3$ (dotted lines), one on the incipient cluster, the other two on the dual empty sites. The entrances of fjords ⊙ close in the scaling limit. Point (●) is first reached by three independent RW's (red, green, blue), contributing to H^3(●). The hull of the incipient cluster (golden line) avoids the outer frontier of the RW's (thick blue line).

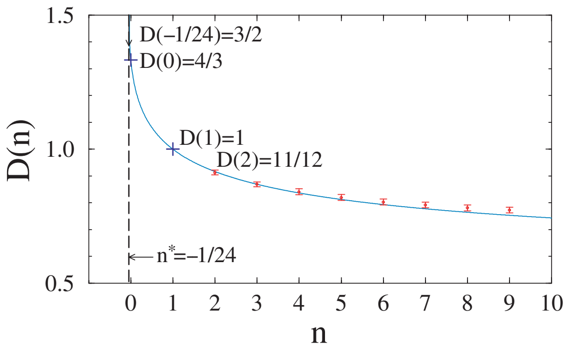

Fig. 5. Universal generalized dimensions $D(n)$ as a function of n, corresponding to the harmonic measure near a percolation cluster, or to self-avoiding or random walks, and comparison with the numerical data obtained by Meakin *et al.* (1988) for percolation.

same, and one of the main conclusions of this study is that *the external frontiers of a planar Brownian motion, or of a critical percolation cluster are identical to a critical self-avoiding walk, with a Hausdorff dimension* $D = \frac{4}{3}$. As we have seen, this fact is linked to the presence of a single universal conformal field theory (with a vanishing central charge $c = 0$), and to the underlying presence of quantum gravity, which organizes the associated conformal dimensions. Note that in a recent work, Smirnov[42] proved that critical site percolation on the triangular lattice has a conformally invariant scaling limit, and that the discrete cluster interfaces (hulls) converge to the same stochastic Löwner evolution process as the one involved for Brownian paths, opening the way to a rigorous derivation of percolation exponents,[43, 44] previously derived in physics.[17, 55, 63, 38]

6. GENERAL CONFORMALLY SCALING CURVES AND HIGHER MULTIFRACTALITY

In the next sections, we present the main description of multifractal functions results in a universal way. For multiple simple curves, we also define the higher multifractal spectra, depending on an arbitrary number of α-variables. We then proceed with their derivation from conformal field theory and quantum gravity. The geometrical findings are described in details, including well-known cases (Ising clusters, $Q = 4$ Potts model). Finally, some geometric duality properties for the external boundaries in $O(N)$ and Potts models are explained. We also make explicit the relation between a conformally invariant scaling curve with CFT central charge c,[34] and the stochastic Löwner process SLE_κ.[39–41]

Consider a single (conformally invariant) critical random cluster, generically called $\mathscr{C}$. Let $H(z)$ be the potential at exterior point $z \in \mathbb{C}$, with Dirichlet boundary conditions $H(w \in \partial\mathscr{C}) = 0$ on the outer (simply connected) boundary $\partial\mathscr{C}$ of $\mathscr{C}$, (or frontier $\mathscr{F} \equiv \partial\mathscr{C}$) , and $H(w) = 1$ on a circle "at ∞," i.e., of a large radius scaling like the average size R of $\mathscr{C}$. From a well-known theorem due to Kakutani,[65] $H(z)$ is identical to the *harmonic measure*, i.e, the probability that a random walker (more precisely, a Brownian motion) launched from z, escapes to ∞ without having hit $\mathscr{C}$. The multifractal formalism[11–14] characterizes subsets $\partial\mathscr{C}_\alpha$ of boundary sites by a Hölder exponent α, and a Hausdorff dimension $f(\alpha) = \dim(\partial\mathscr{C}_\alpha)$, such that their potential locally scales as

$$H(z \to w \in \partial\mathscr{C}_\alpha) \approx (|z - w|/R)^\alpha, \tag{87}$$

in the scaling limit $a \ll r = |z - w| \ll R$, with a the underlying lattice constant. In 2D the *complex* potential $\varphi(z)$ (such that the electrostatic potential $H(z)=\Im\varphi(z)$ and field $|\mathbf{E}(z)|=|\varphi'(z)|$) reads for a *wedge* of angle θ, centered at w:

$$\varphi(z)=(z-w)^{\pi/\theta}. \tag{88}$$

By Eq. (87) a Hölder exponent α thus defines a local equivalent "electrostatic" angle $\theta=\pi/\alpha$, and the MF dimension $\hat{f}(\theta)$ of the boundary subset with such θ is

$$\hat{f}(\theta)=f(\alpha=\pi/\theta). \tag{89}$$

Of special interest are the moments of H, averaged over all realizations of $\mathscr{C}$, and defined in a formal way as

$$\mathscr{Z}_n=\left\langle \sum_{z\,\in\,\partial\mathscr{C}(r)} H^n(z)\right\rangle, \tag{90}$$

where points $z\in\partial\mathscr{C}(r)$ form a discrete set shifted a distance r outwards from $\partial\mathscr{C}$, and where n can be a real number. In the scaling limit, one expects these moments to scale as

$$\mathscr{Z}_n\approx(r/R)^{\tau(n)}, \tag{91}$$

where the multifractal scaling exponents $\tau(n)$ vary in a non-linear way with n;[11–14] they obey the symmetric Legendre transform $\tau(n)+f(\alpha)=\alpha n$, with $n=f'(\alpha)$, $\alpha=\tau'(n)$. From Gauss's theorem[24] $\tau(1)=0$. As noted above, because of the ensemble average (90), values of $f(\alpha)$ can become negative for some domains of α.[22, 24]

Now, we consider the specific case where the fractal set $\mathscr{C}$ is a (conformally invariant) *simple scaling curve*, that is, it does not contain double points. The frontier $\partial\mathscr{C}$ is thus identical with the set itself:

$$\partial\mathscr{C}=\mathscr{C}. \tag{92}$$

Each point of the curve can then be reached from infinity, and we can address the more refined question of the simultaneous behavior of the potential on both sides of the curve. Specifically, the potential H scales as

$$H_+(z\to w^+\in\partial\mathscr{C}_{\alpha,\,\alpha'})\approx|z-w|^{\alpha}, \tag{93}$$

when approaching w on one side of the scaling curve, together with the scaling

$$H_-(z \to w^- \in \partial \mathscr{C}_{\alpha, \alpha'}) \approx |z-w|^{\alpha'}, \tag{94}$$

on the other side. We can then generalize the multifractal formalism to characterize subsets $\mathscr{C}_{\alpha, \alpha'}$ of boundary sites w by two Hölder exponents α, α' such that the potential near w locally scales on the two sides of $\mathscr{C}$ as in Eqs. (93) and (94). This subset is characterized by a Hausdorff dimension $f_2(\alpha, \alpha') = \dim(\mathscr{C}_{\alpha, \alpha'})$. The standard multifractal spectrum $f(\alpha)$ is then recovered as the supremum:

$$f(\alpha) = \sup_{\alpha'} f_2(\alpha, \alpha'). \tag{95}$$

As above, one can also define two equivalent "electrostatic" angles from the Hölder exponents α, α', as $\theta = \pi/\alpha$, $\theta' = \pi/\alpha'$ and the MF dimension $\hat{f}_2(\theta, \theta')$ of the boundary subset with such θ, θ' is then

$$\hat{f}_2(\theta, \theta') = f_2(\alpha = \pi/\theta, \alpha' = \pi/\theta'). \tag{96}$$

Define the harmonic measure $\mathrm{H}(w)$ as the probability that a random walker, launched from infinity, *first* hits the frontier $\mathscr{C}$ at point $w \in \mathscr{C}$. A covering of $\mathscr{C}$ by balls $\mathscr{B}(w, r)$ of radius r is centered at points $w \in \mathscr{C}/\{r\}$, forming a discrete subset $\mathscr{C}/\{r\}$ of $\mathscr{C}$. Let $\mathrm{H}(\mathscr{C} \cap \mathscr{B}(w, r))$ be the harmonic measure of the intersection of $\mathscr{C}$ and the ball $\mathscr{B}(w, r)$. The double multifractal spectrum will be computed from the double moments of the harmonic measure on *both* sides of the random fractal curve. Let us define:

$$\mathscr{Z}_{n, n'} = \left\langle \sum_{w \in \mathscr{C}/\{r\}} [\mathrm{H}_+(w)]^n \, [\mathrm{H}_-(w)]^{n'} \right\rangle, \tag{97}$$

where $\mathrm{H}_+(w) \equiv \mathrm{H}_+(\mathscr{C} \cap \mathscr{B}(w, r))$ and $\mathrm{H}_-(w) \equiv \mathrm{H}_-(\mathscr{C} \cap \mathscr{B}(w, r))$ are respectively the harmonic measures on the "left" or "right" sides of the random fractal. These double moments have a multifractal scaling behavior

$$\mathscr{Z}_{n, n'} \approx (r/R)^{\tau_2(n, n')}, \tag{98}$$

where the exponent $\tau_2(n, n')$ now depends on two moment orders n, n'. The generalization of the usual Legendre transform of multifractal formalism Eq. (6) reads

$$\alpha = \frac{\partial \tau_2}{\partial n}(n, n'), \qquad \alpha' = \frac{\partial \tau_2}{\partial n'}(n, n'),$$
$$f_2(\alpha, \alpha') = \alpha n + \alpha' n' - \tau_2(n, n'), \tag{99}$$
$$n = \frac{\partial f_2}{\partial \alpha}(\alpha, \alpha'), \qquad n' = \frac{\partial f_2}{\partial \alpha'}(\alpha, \alpha').$$

From definition (97) and Eq. (98), we recover the one-sided multifractal spectrum,

$$\tau(n) = \tau_2(n, n' = 0). \tag{100}$$

Putting the value $n' = 0$ in the Legendre transform Eq. (99), we obtain the identity (95), as we must.

More generally, one can consider a star configuration $\mathscr{S}_m$ of a number m, $m \geqslant 2$, of *similar simple scaling paths*, all originating at the same vertex w. The higher moments $\mathscr{Z}_{n_1, n_2, \dots, n_m}$ can then be defined as

$$\mathscr{Z}_{n_1, n_2, \dots, n_m} = \left\langle \sum_{w \in \mathscr{S}_m} [\mathrm{H}_1(w)]^{n_1} [\mathrm{H}_2(w)]^{n_2} \cdots [\mathrm{H}_m(w)]^{n_m} \right\rangle, \tag{101}$$

where

$$\mathrm{H}_i(w) \equiv \mathrm{H}_i(\mathscr{C} \cap \mathscr{B}(w, r))$$

is the harmonic measure (or, equivalently, local potential at distance r) in the ith sector of radius located between paths i and $i+1$, with $i = 1, \dots, m$, and by periodicity $m+1 \equiv 1$. These higher moments have a multifractal scaling behavior

$$\mathscr{Z}_{n_1, n_2, \dots, n_m} \approx (r/R)^{\tau_m(n_1, n_2, \dots, n_m)}, \tag{102}$$

where the exponent $\tau_m(n_1, n_2, \dots, n_m)$ now depends on the set of moment orders $n_1, n_2, \dots, n_m$. The generalization of the usual Legendre transform of multifractal formalism Eq. (6) now involves a higher multifractal function $f_m(\alpha_1, \alpha_2, \dots, \alpha_m)$, depending on m local exponents α_i:

$$\alpha_i = \frac{\partial \tau_m}{\partial n_i}(\{n_i\}),$$
$$f_m(\{\alpha_i\}) = \sum_{i=1}^{m} \alpha_i n_i - \tau_m(\{n_i\}), \tag{103}$$
$$n_i = \frac{\partial f_m}{\partial \alpha_i}(\{\alpha_j\}).$$

At this point, a caveat is in order. The reader may wonder about the meaning of the sum over point w in (101), if there is only one such m-vertex in a star! This formal notation is kept for consistency with the $m=2$ case, and is meant to indicate that exponents $\tau_m(n_1, n_2,..., n_m)$ are calculated with inclusion of the Hausdorff dimension D_m, associated with the star center; in these notations $D_m = -\tau_m(0, 0,..., 0) = \sup_{\{\alpha_i\}} f_m(\{\alpha_i\})$, and it becomes negative for m high enough (see Section 8 below). One can define shifted exponents $\tilde{\tau}_m \equiv \tau_m + D_m = \tau_m(n_1, n_2,..., n_m) - \tau_m(0, 0,..., 0)$, which correspond to a different normalization, and describe the scaling of local averages

$$\langle [\mathrm{H}_1(w)]^{n_1} [\mathrm{H}_2(w)]^{n_2} \cdots [\mathrm{H}_m(w)]^{n_m} \rangle \approx (r/R)^{\tilde{\tau}_m(n_1, n_2,..., n_m)}. \tag{104}$$

By Legendre transform (103) these exponents give the *subtracted* spectrum $f_m(\{\alpha_i\}) - \sup_{\{\alpha_i\}} f_m(\{\alpha_i\})$ directly. The latter has an immediate physical meaning: the probability $P(\{\alpha_i\})$ to find a set of local singularity exponents $\{\alpha_i\}$ in the m sectors of an m-arm star scales as:

$$P_m(\{\alpha_i\}) \propto R^{f_m(\{\alpha_i\})} / R^{\sup f_m}. \tag{105}$$

From definition (101) and Eq. (102), we get the lower $(m-1)$-multifractal spectrum as

$$\tau_m^{[m-1]}(n_1, n_2,..., n_{m-1}) = \tau_m(n_1, n_2,..., n_{m-1}, n_m = 0). \tag{106}$$

In these exponents, the subscript m stays unchanged since it counts the number of arms of the star, while the potential is evaluated only at $m-1$ sectors among the m possible ones. More generally, one can define exponents

$$\tau_m^{[p]}(n_1, n_2,..., n_p) = \tau_m(n_1, n_2,..., n_p;\, n_{p+1} = 0,..., n_m = 0),$$

where p takes any value in $1 \leqslant p \leqslant m$. Note that according to the commutativity of the star algebra for exponents between mutually-avoiding paths (see Eq. (43) and below), the result does not depend on the choice of the p sectors among m. Putting the value $n_m = 0$ in the Legendre transform Eq. (103), we obtain the identity:

$$f_m^{[m-1]}(\alpha_1,..., \alpha_{m-1}) = \sup_{\alpha_m} f_m(\alpha_1, \alpha_2,..., \alpha_m). \tag{107}$$

Note that the usual $f(\alpha)$ spectrum is in these notations $f_2^{[1]}(\alpha)$.

7. CONFORMAL INVARIANCE AND QUANTUM GRAVITY

Let us now follow the main lines of the derivation of exponents $\tau(n)$, hence $f(\alpha)$, or, more generally, $\tau_m(n_1, n_2,..., n_m)$ and $f_m(\alpha_1, \alpha_2,..., \alpha_m)$ via generalized *conformal invariance*. By definition of the H-measure, *n independent* RW's, or Brownian paths $\mathscr{B}$ in the scaling limit, starting at the same point a distance r away from the cluster's hull's frontier $\partial\mathscr{C}$, and diffusing without hitting $\partial\mathscr{C}$, give a geometric representation of the nth moment, H^n, in Eq. (47) for *n integer*. Convexity yields analytic continuation for arbitrary n's. Let us recall the notation $A \wedge B$ for two random sets conditioned to traverse, *without mutual intersection*, the annulus $\mathscr{D}(r, R)$ from the inner boundary circle of radius r to the outer one at distance R, and $A \vee B$ for two *independent*, thus possibly intersecting, sets.[29] With this notation, the "probability" (actually the associated grand canonical partition function) that the Brownian paths and cluster are in a configuration $\partial\mathscr{C} \wedge (\vee \mathscr{B})^n \equiv \partial\mathscr{C} \wedge n$, is expected to scale for $R/r \to \infty$ as

$$\mathscr{P}_R(\partial\mathscr{C} \wedge n) \approx (r/R)^{x(n)}, \tag{108}$$

where the scaling exponent $x(n)$ depends on n. In terms of definition (108), the harmonic measure moments (90) simply scale as $\mathscr{Z}_n \approx R^2 \mathscr{P}_R(\partial\mathscr{C} \wedge n)$,[24, 29] which, combined with Eq. (48), leads to

$$\tau(n) = x(n) - 2. \tag{109}$$

To calculate these exponents, we use the fundamental mapping of the conformal field theory in the *plane* $\mathbb{R}^2$, describing a critical statistical system, to the CFT on a fluctuating abstract random Riemann surface, i.e., in presence of *quantum gravity*.[45, 50, 49] Two universal functions U, and V, which now depend on the central charge c of the CFT, describe this map:

$$U(x) = x \frac{x-\gamma}{1-\gamma}, \qquad V(x) = \frac{1}{4} \frac{x^2-\gamma^2}{1-\gamma}, \tag{110}$$

with

$$V(x) \equiv U\left(\tfrac{1}{2}(x+\gamma)\right). \tag{111}$$

The parameter γ is the *string susceptibility exponent* of the random 2D surface (of genus zero), bearing the CFT of central charge c;[45] γ is the solution of

$$c = 1 - 6\gamma^2(1-\gamma)^{-1}, \qquad \gamma \leqslant 0. \tag{112}$$

For two arbitrary random sets A, B, with boundary scaling exponents in the *half-plane* $\tilde{x}(A)$, $\tilde{x}(B)$, the scaling exponent $x(A \wedge B)$, as in (108), has the universal structure[29, 30, 34]

$$x(A \wedge B) = 2V[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))], \tag{113}$$

where $U^{-1}(x)$ is the *positive* inverse function of U

$$U^{-1}(x) = \tfrac{1}{2}(\sqrt{4(1-\gamma)\,x + \gamma^2} + \gamma). \tag{114}$$

Note that one has the shift relation

$$U^{-1}(x) = \tfrac{1}{2}V^{-1}(x) + \tfrac{1}{2}\gamma, \tag{115}$$

where

$$V^{-1}(x) = \sqrt{4(1-\gamma)\,x + \gamma^2}\,. \tag{116}$$

$U^{-1}(\tilde{x})$ is, on the random Riemann surface, the boundary scaling dimension corresponding to $\tilde{x}$ in the half-plane $\mathbb{R} \times \mathbb{R}^+$, and the sum of U^{-1} functions in Eq. (113) is a *linear* representation of the product of two "boundary operators" on the random surface, as the condition $A \wedge B$ for two mutually-avoiding sets is purely *topological* there. The sum is mapped back by the function V into the scaling dimensions in $\mathbb{R}^2$.[34]

For the harmonic exponents $x(n) \equiv x(\partial\mathscr{C} \wedge n)$ in (108), we use (113). The *external boundary* exponent $\tilde{x}(\partial\mathscr{C})$ obeys

$$U^{-1}(\tilde{x}) = 1 - \gamma, \tag{117}$$

which we derive either directly, or from Makarov's theorem:

$$\tau'(n=1) = \frac{dx}{dn}(n=1) = 1. \tag{118}$$

The bunch of n independent Brownian paths has simply $\tilde{x}((\vee\mathscr{B})^n) = n$, since $\tilde{x}(\mathscr{B}) = 1$.[29] Thus we obtain

$$x(n) = 2V(1 - \gamma + U^{-1}(n)). \tag{119}$$

This finally gives from (110) (114) for $\tau(n) = x(n) - 2$:[34]

$$\tau(n) = \frac{1}{2}(n-1) + \frac{1}{4}\frac{2-\gamma}{1-\gamma}[\sqrt{4(1-\gamma)\,n + \gamma^2} - (2-\gamma)]. \tag{120}$$

Similar exponents associated with moments later appeared in the context of the *SLE* process (see II in ref. 40; see also ref. 66 for Laplacian random walks).

The Legendre transform is easily performed to yield:

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{1}{2}\frac{2-\gamma}{\sqrt{4(1-\gamma)\,n+\gamma^2}}; \tag{121}$$

$$f(\alpha) = \frac{1}{8}\frac{(2-\gamma)^2}{1-\gamma}\left(3 - \frac{1}{2\alpha-1}\right) - \frac{1}{4}\frac{\gamma^2}{1-\gamma}\,\alpha, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right). \tag{122}$$

Using the identities in terms of central charge c:

$$\begin{aligned} \frac{1}{4}\frac{(2-\gamma)^2}{1-\gamma} &= \frac{25-c}{24} \\ \frac{1}{4}\frac{\gamma^2}{1-\gamma} &= \frac{1-c}{24}, \end{aligned} \tag{123}$$

we find

$$\tau(n) = \frac{1}{2}(n-1) + \frac{25-c}{24}\left(\sqrt{\frac{24n+1-c}{25-c}} - 1\right) \qquad n \in \left[n^* = -\frac{1-c}{24}, +\infty\right).$$

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{25-c}{24n+1-c}}; \tag{124}$$

$$f(\alpha) = \frac{25-c}{48}\left(3 - \frac{1}{2\alpha-1}\right) - \frac{1-c}{24}\,\alpha, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right). \tag{125}$$

This formalism immediately allows generalizations. For instance, in place of n random walks, one can consider a set of n *independent self-avoiding* walks $\mathscr{P}$, which avoid the cluster fractal boundary, except for their common anchoring point. The associated multifractal exponents $x(\partial\mathscr{C} \wedge (\vee\mathscr{P})^n)$ are given by (119), with the argument n in $U^{-1}(n)$ simply replaced by $\tilde{x}((\vee\mathscr{P})^n) = n\tilde{x}(\mathscr{P}) = \frac{5}{8}n$.[29] These exponents govern the universal multifractal behavior of the moments of the probability that a SAW escapes from $\mathscr{C}$. One then gets a spectrum $\bar{f}$ such that $\bar{f}(\bar{\alpha} = \tilde{x}(\mathscr{P})\,\pi/\theta) = f(\alpha = \pi/\theta) = \hat{f}(\theta)$, thus unveiling the *same invariant* underlying wedge distribution as the harmonic measure, (see also ref. 33).

8. HIGHER MULTIFRACTAL SPECTRA

In analogy to Eqs. (109), (119), the exponent $\tau_2(n, n')$ is associated with a scaling dimension $x_2(n, n')$ by

$$\begin{aligned} \tau_2(n, n') &= x_2(n, n') - 2 \\ x_2(n, n') &= 2V[1-\gamma+U^{-1}(n)+U^{-1}(n')]. \end{aligned} \tag{126}$$

The calculation of the double Legendre transform Eq. (99) is as follows. We start with the notation for the total quantum gravity scaling dimension:

$$\delta \equiv 1-\gamma+U^{-1}(n)+U^{-1}(n'). \tag{127}$$

This gives explicitly:

$$\delta = 1+\tfrac{1}{2}\sqrt{4(1-\gamma)\,n+\gamma^2}+\tfrac{1}{2}\sqrt{4(1-\gamma)\,n'+\gamma^2}. \tag{128}$$

Then, we have

$$\alpha = \frac{\partial x_2}{\partial n}(n, n') = 2V'(\delta)\,\frac{\partial \delta}{\partial n} \tag{129}$$

and since

$$V'(x) = \frac{1}{2}\,\frac{x}{1-\gamma},$$

we finally get

$$\alpha = \frac{\delta}{\sqrt{4(1-\gamma)\,n+\gamma^2}}, \qquad \alpha' = \frac{\delta}{\sqrt{4(1-\gamma)\,n'+\gamma^2}}. \tag{130}$$

A useful consequence is the identity

$$\delta = \left[1-\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\alpha'}\right)\right]^{-1}. \tag{131}$$

Equation (130) can be inverted into

$$n = \frac{1}{4(1-\gamma)}\left(\frac{\delta^2}{\alpha^2}-\gamma^2\right) = V\left(\frac{\delta}{\alpha}\right), \tag{132}$$

where use was made of (110) for V. This allows the simple expression of f_2

$$f_2(\alpha, \alpha') = 2 - V(\delta) + \alpha V\left(\frac{\delta}{\alpha}\right) + \alpha' V\left(\frac{\delta}{\alpha'}\right). \tag{133}$$

Reordering the δ terms with use of (110), and recalling identity (131) for δ, finally gives after some calculations the explicit formulae

$$f_2(\alpha, \alpha') = \frac{25-c}{12} - \frac{1}{2(1-\gamma)}\left[1 - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\alpha'}\right)\right]^{-1} - \frac{1-c}{24}(\alpha+\alpha'), \tag{134}$$

$$\alpha = \frac{1}{\sqrt{4(1-\gamma)\,n+\gamma^2}}\left[1 + \frac{1}{2}\left(\sqrt{4(1-\gamma)\,n+\gamma^2} + \sqrt{4(1-\gamma)\,n'+\gamma^2}\right)\right], \tag{135}$$

where the central charge c and the parameter γ are related by Eqs. (123). This doubly multifractal spectrum possesses the desired properties, like $\sup_{\alpha'} f_2(\alpha, \alpha') = f(\alpha)$, where $f(\alpha)$ is (122) above.

This double multifractality can be generalized to higher ones by considering *star configurations* made of m simple scaling paths all originating at the same vertex, as in (101), with the following *poly-multifractal* results. The m-order case will be given by

$$\tau_m(n_1, n_2,..., n_m) = x_m(n_1, n_2,..., n_m) - 2$$
$$x_m(n_1, n_2,..., n_m) = 2V[\tilde{\Delta}_m + U^{-1}(n_1) + U^{-1}(n_2) + \cdots + U^{-1}(n_m)].$$

Here $\tilde{\Delta}_m$ is the quantum gravity boundary scaling dimension of the m-star $\mathcal{S}_m$ made of m (simple) scaling paths. According to the star algebra we have:

$$\tilde{\Delta}_m = m\,U^{-1}(\tilde{x}_1) = \frac{m}{2}\,U^{-1}(\tilde{x}_2) = m\,\frac{1-\gamma}{2}, \tag{136}$$

whre $\tilde{x}_2 \equiv \tilde{x}$ is the boundary scaling dimension of a scaling path, i.e., a 2-star, already considered in Eq. (117), and such that $U^{-1}(\tilde{x}) = 1-\gamma$. We therefore arrive at a total (boundary) quantum scaling dimension

$$\delta_m = m\,\frac{1-\gamma}{2} + \sum_{i=1}^{m} U^{-1}(n_i), \tag{137}$$

such that

$$x_m(n_1, n_2,..., n_m) = 2V(\delta_m). \tag{138}$$

Using the shift identity (see (115))

$$U^{-1}(n) = \frac{\gamma}{2} + \frac{1}{2} V^{-1}(n),$$

where we recall that

$$V^{-1}(n) = \sqrt{4(1-\gamma)\, n + \gamma^2}\,,$$

we also have

$$\delta_m = \frac{m}{2} + \frac{1}{2} \sum_{i=1}^{m} V^{-1}(n_i). \tag{139}$$

The multiple Legendre transform (103) is performed as above for the case $m=2$. We have

$$\alpha_i = \frac{\partial x_m}{\partial n_i}(\{n_j\}) = 2V'(\delta_m)\, \frac{\partial \delta_m}{\partial n_i} = V'(\delta_m)\, [V^{-1}(n_i)]', \tag{140}$$

so we get

$$\alpha_i = \frac{\delta_m}{\sqrt{4(1-\gamma)\, n_i + \gamma^2}} = \frac{\delta_m}{V^{-1}(n_i)}, \tag{141}$$

or, equivalently

$$V^{-1}(n_i) = \frac{\delta_m}{\alpha_i}, \tag{142}$$

inverted into

$$n_i = V\left(\frac{\delta_m}{\alpha_i}\right). \tag{143}$$

One gets from Eqs. (139) and (142)

$$\delta_m = \left(1 - \frac{1}{2} \sum_{i=1}^{m} \alpha_i^{-1}\right)^{-1}. \tag{144}$$

This allows the simple expression of f_m

$$f_m(\{\alpha_i\}) = 2 - V(\delta_m) + \sum_{i=1}^{m} \alpha_i V\left(\frac{\delta_m}{\alpha_i}\right). \tag{145}$$

Reordering the δ_m terms with use of (110), and recalling identity (144) for δ_m, finally gives after some calculations the explicit formulae

$$f_m(\{\alpha_{i=1,\dots,m}\}) = 2+\frac{\gamma^2}{2(1-\gamma)}-\frac{1}{8(1-\gamma)}\,m^2\left(1-\frac{1}{2}\sum_{i=1}^{m}\alpha_i^{-1}\right)^{-1}-\frac{\gamma^2}{4(1-\gamma)}\sum_{i=1}^{m}\alpha_i, \tag{146}$$

with

$$\alpha_i = \frac{1}{\sqrt{4(1-\gamma)\,n_i+\gamma^2}}\left(\frac{m}{2}+\frac{1}{2}\sum_{j=1}^{m}\sqrt{4(1-\gamma)\,n_j+\gamma^2}\right). \tag{147}$$

Substituting expressions (123) gives in terms of c

$$f_m(\{\alpha_{i=1,\dots,m}\}) = \frac{25-c}{12}-\frac{1}{8(1-\gamma)}\,m^2\left(1-\frac{1}{2}\sum_{i=1}^{m}\alpha_i^{-1}\right)^{-1}-\frac{1-c}{24}\sum_{i=1}^{m}\alpha_i. \tag{148}$$

The domain of definition of the poly-multifractal function f is independent of c and given by

$$1-\frac{1}{2}\sum_{i=1}^{m}\alpha_i^{-1}\geqslant 0, \tag{149}$$

as verified by Eq. (147). The two-sided case (134) above is recovered for $m=2$, while the self-avoiding walk case (78) is recovered for $\gamma=-1/2$, $c=0$. Notice that the case $f_{m=1}(\alpha_1)$ corresponds to the potential in the vicinity of the tip of a conformally invariant scaling path, and differs from the usual $f(\alpha)=\sup_{\alpha'} f_2(\alpha,\alpha'))$ spectrum, which describes the potential on one side of the scaling path.

We can also substitute equivalent "electrostatic" angles $\theta_i=\pi/\alpha_i$ for the variables α_i. This gives a new distribution:

$$\begin{aligned}\hat{f}_m(\{\theta_{i=1,\dots,m}\}) &\equiv f_m(\{\alpha_{i=1,\dots,m}\})\\ &= 2+\frac{\gamma^2}{2(1-\gamma)}-\frac{\gamma^2}{4(1-\gamma)}\sum_{i=1}^{m}\frac{\pi}{\theta_i}-\frac{1}{8(1-\gamma)}\,m^2\left(1-\frac{1}{2\pi}\sum_{i=1}^{m}\theta_i\right)^{-1}.\end{aligned} \tag{150}$$

The domain of definition of distribution $\hat{f}_m$ is the image of domain (149) in θ-variables:

$$\sum_{i=1}^{m}\theta_i\leqslant 2\pi. \tag{151}$$

The total *electrostatic* angle is thus less than 2π, which simply accounts for the electrostatic screening of local wedges by fractal randomness, as expected.

The maximum of f_m or $\hat{f}_m$ is by construction obtained for $n_i = 0$, $\forall i = 1,..., m$. Equation (147) gives the values of singularity exponents $\hat{\alpha}_i$ at the maximum of f_m:

$$\hat{\alpha}_i = \frac{\pi}{\hat{\theta}_i} = \frac{m}{2}\left(1 - \frac{1}{\gamma}\right)^{-1}, \qquad \forall i = 1,..., m, \tag{152}$$

corresponding to a maximum value of f_m or $\hat{f}_m$:

$$\begin{aligned} \sup f_m = f_m(\{\hat{\alpha}_{i=1,...,m}\}) = \hat{f}_m(\{\hat{\theta}_{i=1,...,m}\}) &= 2 - 2V(\tilde{\Delta}_m) \\ &= 2 + \frac{\gamma^2}{2(1-\gamma)} - \frac{1}{8(1-\gamma)} m^2. \end{aligned}$$

The interpretation of the poly-multifractal spectrum can be understood as follows. The probability $P(\{\alpha_i\}) \equiv \hat{P}(\{\theta_i\})$ to find a set of local singularity exponents $\{\alpha_i\}$ or equivalent angles $\{\theta_i\}$ in the m sectors of an m-arm star is given by the ratio

$$P_m(\{\alpha_i\}) \propto R^{f_m(\{\alpha_i\})} / R^{\sup f_m} \tag{153}$$

of the respective number of configurations to the total one. We therefore arrive at a probability, here written in terms of the equivalent electrostatic angles:

$$\hat{P}_m(\{\theta_i\}) \propto R^{\hat{f}_m(\{\theta_i\}) - \hat{f}_m(\{\hat{\theta}_i\})}, \tag{154}$$

$$\hat{f}_m(\{\theta_i\}) - \hat{f}_m(\{\hat{\theta}_i\}) = -\frac{1}{8(1-\gamma)} m^2 \left(\frac{2\pi}{\sum_{i=1}^m \theta_i} - 1\right)^{-1} - \frac{\gamma^2}{4(1-\gamma)} \sum_{i=1}^m \frac{\pi}{\theta_i}. \tag{155}$$

For a large scaling star, the dominant set of singularity exponents $\{\hat{\alpha}_i\}$, or wedge angles $\{\hat{\theta}_i\}$, is thus given by the symmetric set of values (152).

9. ANALYSIS OF MULTIFRACTAL DIMENSIONS AND SPECTRA

Let us collect the results for the one-sided functions $\tau(n)$, $D(n)$, and $f(\alpha)$. Each conformally invariant random system is labelled by its *central* *charge* c, $c \leqslant 1$. The multifractal exponents $\tau(n)$ and generalized dimensions $D(n)$ of a simply connected CI boundary are then:

$$\tau(n) = \frac{1}{2}(n-1) + \frac{25-c}{24}\left(\sqrt{\frac{24n+1-c}{25-c}} - 1\right), \tag{156}$$

$$D(n) = \frac{\tau(n)}{n-1} = \frac{1}{2} + \left(\sqrt{\frac{24n+1-c}{25-c}} + 1\right)^{-1}, \qquad n \in \left[n^* = -\frac{1-c}{24}, +\infty\right); \tag{157}$$

after a Legendre transform:

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{25-c}{24n+1-c}}; \tag{158}$$

$$f(\alpha) = \frac{25-c}{48}\left(3 - \frac{1}{2\alpha - 1}\right) - \frac{1-c}{24}\alpha, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right). \tag{159}$$

It is interesting to note that the general multifractal function (159) possesses the invariance property (63), since it can also be written as

$$f(\alpha) - \alpha = \frac{25-c}{24}\left[1 - \frac{1}{2}\left(2\alpha - 1 + \frac{1}{2\alpha - 1}\right)\right]. \tag{160}$$

Notice that the generalized dimensions $D(n)$ satisfy, for any c, $\tau'(n=1) = D(n=1) = 1$, or equivalently $f(\alpha = 1) = 1$, i.e., *Makarov's theorem*,[36] valid for any simply connected boundary curve. From (157), (158) we also remark a fundamental relation, independent of c:

$$3 - 2D(n) = 1/\alpha = \theta/\pi. \tag{161}$$

We also have the *superuniversal* bounds: $\forall c$, $\forall n$, $\frac{1}{2} = D(\infty) \leqslant D(n) \leqslant D(n^*) = \frac{3}{2}$, hence $0 \leqslant \theta \leqslant 2\pi$. We arrive at the geometrical multifractal distribution of wedges θ along the boundary:

$$\hat{f}(\theta) = f\left(\frac{\pi}{\theta}\right) = \frac{\pi}{\theta} - \frac{25-c}{12}\frac{(\pi-\theta)^2}{\theta(2\pi-\theta)}. \tag{162}$$

Remarkably enough, the second term also describes the contribution by a wedge to the density of electromagnetic modes in a cavity.[66] The simple shift in (162), $25 \to 25 - c$, from the $c = 0$ case to general values of c can then be related to results of conformal invariance in a wedge.[67] The partition function for the two sides of a wedge of angle θ and size R, in a CFT of central charge c, indeed scales as[68]

$$\hat{\mathscr{Z}}(\theta) \approx R^{-c(\pi-\theta)^2/12\,\theta(2\pi-\theta)}. \tag{163}$$

Thus, one can view the c dependance of result (162) as a renormalization of the number of sites with wedge angle θ along a self-avoiding scaling curve by a partition factor $[\hat{\mathscr{Z}}(\theta)]^{-1}$, representing the presence of a c-CFT along such wedges.

The maximum of $f(\alpha)$ corresponds to $n=0$, and gives the dimension D_{EP} of the support of the measure, i.e., the *accessible* or *external perimeter* as:

$$D_{\mathrm{EP}} = \sup_{\alpha} \; f(\alpha) = f(\alpha(n=0)) \tag{164}$$

$$= D(0) = \frac{3-2\gamma}{2(1-\gamma)} = \frac{3}{2} - \frac{1}{24}\sqrt{1-c}\,(\sqrt{25-c}-\sqrt{1-c}). \tag{165}$$

This corresponds to a *typical exponent*

$$\hat{\alpha} = \alpha(0) = 1 - \frac{1}{\gamma} = \left(\frac{1}{12}\sqrt{1-c}\,(\sqrt{25-c}-\sqrt{1-c})\right)^{-1} = (3-2D_{\mathrm{EP}})^{-1}. \tag{166}$$

This corresponds to a *typical wedge angle*

$$\hat{\theta} = \pi/\hat{\alpha} = \pi(3-2D_{\mathrm{EP}}). \tag{167}$$

In analogy to the probability (153) for multiple angles, the probability $P(\alpha)$ to find a singularity exponent α or, equivalently, $\hat{P}(\theta)$ to find an equivalent opening angle θ along the frontier is

$$P(\alpha) = \hat{P}(\theta) \propto R^{f(\alpha)-f(\hat{\alpha})}. \tag{168}$$

Using the values found above, this probability can be recast as (see also ref. 33)

$$P(\alpha) = \hat{P}(\theta) \propto \exp\left[-\frac{1}{24}\ln R\left(\sqrt{1-c}\,\sqrt{\omega} - \frac{\sqrt{25-c}}{2\sqrt{\omega}}\right)^2\right], \tag{169}$$

where

$$\omega = \alpha - \frac{1}{2} = \frac{\pi}{\theta} - \frac{1}{2}.$$

The multifractal functions $f(\alpha)-\alpha=\hat{f}(\theta)-\frac{\pi}{\theta}$ are *invariant* when taken upon the substitution of primed variables given by

$$2\pi=\theta+\theta'=\frac{\pi}{\alpha}+\frac{\pi}{\alpha'}, \tag{170}$$

this corresponds to the complementary domain of the wedge θ. This condition reads also $D(n)+D(n')=2$. This basic symmetry, first observed and studied in ref. 60 for the $c=0$ result of ref. 29, is valid for *any* conformally invariant boundary.

The multifractal exponents $\tau(n)$ (Fig. 6) or generalized dimensions $D(n)$ (Fig. 7) appear as quite similar for various values of c, and a numerical simulation would hardly distinguish the different universality classes, while the $f(\alpha)$ functions, as we shall see, do distinguish these classes, especially for negative n, i.e., large α. In Fig. 8 are displayed the multifractal functions f, Eq. (159), corresponding to various values of $-2\leqslant c\leqslant 1$, or, equivalently, to a number of components $N\in[0,2]$, and $Q\in[0,4]$ in the $O(N)$ or Potts models (see below).

The singularity at $\alpha=\frac{1}{2}$, or $\theta=2\pi$, in the multifractal functions f, or $\hat{f}$, corresponds to boundary points with a needle local geometry, and Beurling's theorem[58] indeed insures the Hölder exponents α to be bounded below by $\frac{1}{2}$. This corresponds to large values of n, where, asymptotically, for *any* universality class,

$$\forall c,\quad \lim_{n\to\infty} D(n)=\tfrac{1}{2}. \tag{171}$$

The right branch of $f(\alpha)$ has a linear asymptote

$$\lim_{\alpha\to\infty} f(\alpha)/\alpha=n^*=-(1-c)/24. \tag{172}$$

The *limit* multifractal spectrum is obtained for $c=1$, which exhibits an *exact* example of a *left-sided* MF spectrum, with an asymptote $f(\alpha\to\infty, c=1)\to\frac{3}{2}$ (Fig. 8). It corresponds to singular boundaries where $\hat{f}(\theta\to 0, c=1)=\frac{3}{2}=D_{\mathrm{EP}}$, i.e., where the external perimeter is everywhere dominated by "*fjords*," with typical angle $\hat{\theta}=0$. It is tempting to call it the "Ultimate Norway."

The $\alpha\to\infty$ behavior corresponds to moments of lowest order $n\to n^*$, where $D(n)$ reaches its maximal value: $\forall c$, $D(n^*)=\frac{3}{2}$, common to *all* simply connected, conformally invariant, boundaries. This describes almost inaccessible sites: define $\mathcal{N}(H)$ as the number of boundary sites having a given

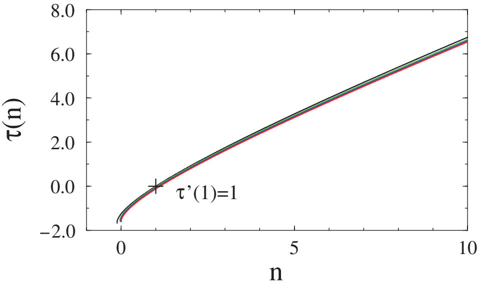

Fig. 6. Universal multifractal exponents $\tau(n)$. The curves are indexed by the central charge c, and correspond to the same colors as in Fig. 8 below: (black: 2D spanning trees ($c=-2$); green: self-avoiding or random walks, and percolation ($c=0$); blue: Ising clusters or $Q=2$ Potts clusters ($c=\frac{1}{2}$); red: $N=2$ loops, or $Q=4$ Potts clusters ($c=1$). The curves are almost indistinguishable at the scale shown.

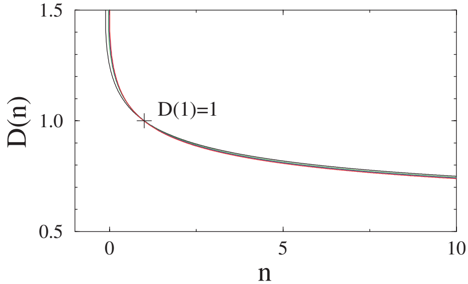

Fig. 7. Universal generalized dimensions $D(n)$. The curves are indexed by the same colors as in Fig. 8 below but are almost indistinguishable at the scale shown.

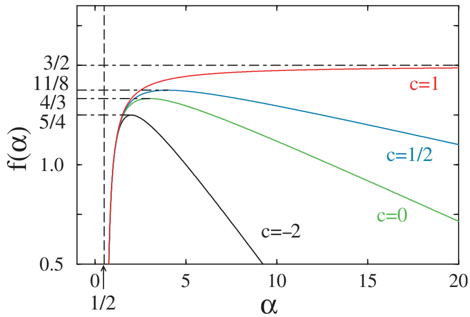

Fig. 8. Universal harmonic multifractal spectra $f(\alpha)$. The curves are indexed by the central charge c, and correspond respectively to: 2D spanning trees ($c=-2$); self-avoiding or random walks, and percolation ($c=0$); Ising clusters or $Q=2$ Potts clusters ($c=\frac{1}{2}$); $N=2$ loops, or $Q=4$ Potts clusters ($c=1$). The maximal dimensions are those of the *accessible* frontiers. The left branches of the various $f(\alpha)$ curves are largely indistinguishable, while their right branches split for large α, corresponding to negative values of n.

probability H to be hit by a RW starting at infinity; the MF formalism yields, for $H \to 0$, a power law behavior

$$\mathcal{N}(H)|_{H\to 0} \approx H^{-(1+n^*)} \tag{173}$$

with an exponent

$$1+n^* = \frac{23+c}{24} < 1. \tag{174}$$

Some particular cases are worth considering. An *Ising* cluster possesses a multifractal spectrum with respect to the harmonic measure ($c = \frac{1}{2}$):

$$\tau(n) = \frac{1}{2}(n-1) + \frac{7}{48}(\sqrt{48n+1}-7), \tag{175}$$

$$f(\alpha) = \frac{49}{96}\left(3 - \frac{1}{2\alpha-1}\right) - \frac{\alpha}{48}, \qquad \alpha \in \left(\frac{1}{2}, +\infty\right), \tag{176}$$

with a dimension of the accessible perimeter

$$D_{\mathrm{EP}} = \sup_{\alpha} f(\alpha, c = \tfrac{1}{2}) = \tfrac{11}{8}. \tag{177}$$

The $Q = 4$ Potts model provides an interesting example of a *left-handed* multifractal spectrum ($c = 1$)

$$\tau(n) = \frac{1}{2}(n-1) + \sqrt{n} - 1, \tag{178}$$

$$f(\alpha) = \frac{1}{2}\left(3 - \frac{1}{2\alpha-1}\right), \qquad \alpha \in \left(\frac{1}{2}, +\infty\right), \tag{179}$$

with accessible sites forming a set of Hausdorff dimenson

$$D_{\mathrm{EP}} = \sup_{\alpha} f(\alpha, c = 1) = \tfrac{3}{2}, \tag{180}$$

which is also the *maximal* value common to all multifractal generalized dimensions $D(n) = \frac{1}{n-1}\tau(n)$. Notice that the external perimeter which bears the electrostatic charge is a *simple* curve, i.e., a curve without double points, a self-avoiding or *simple* path. We therefore arrive at the striking conclusion that in the plane, a conformally invariant scaling curve which is self-avoiding has a Hausdorff dimension at most equal to $D_{\mathrm{EP}} = 3/2$.[34]

The corresponding $Q=4$ Potts frontier, while still possessing a set of double points of dimension 0, actually develops a logarithmically growing number of double points.[69]

10. GEOMETRIC DUALITY IN $O(N)$ AND POTTS CLUSTER FRONTIERS

The $O(N)$ model partition function is that of a gas $\mathcal{G}$ of self- and mutually-avoiding *loops* on a given lattice, e.g., the hexagonal lattice[18]: $Z_{O(N)} = \sum_{\mathcal{G}} K^{\mathcal{N}_B} N^{\mathcal{N}_P}$, with K and N two fugacities, associated respectively with the total number of occupied bonds $\mathcal{N}_B$, and with the total number $\mathcal{N}_P$ of loops, i.e., polygons drawn on the lattice. For $N \in [-2, 2]$, this model possesses a critical point (CP), K_c, while the whole *"low-temperature"* (low-T) phase, i.e., $K_c < K$, has critical universal properties, where the loops are *denser* that those at the critical point.[18]

The partition function of the Q-state Potts model on, e.g., the square lattice, with a second order critical point for $Q \in [0, 4]$, has a Fortuin–Kasteleyn representation *at* the CP: $Z_{\text{Potts}} = \sum_{\cup(\mathcal{C})} Q^{\frac{1}{2}\mathcal{N}_P}$, where the configurations $\cup(\mathcal{C})$ are those of unions of clusters on the square lattice, with a total number $\mathcal{N}_P$ of polygons encircling all clusters, and filling the medial square lattice of the original lattice.[18, 17] Thus the critical Potts model becomes a *dense* loop model, with loop fugacity $N = Q^{\frac{1}{2}}$, while one can show that its *tricritical* point with site dilution corresponds to the $O(N)$ CP.[70] The $O(N)$ and Potts models thus possess the same "Coulomb gas" representations[18, 17, 70]:

$$N = \sqrt{Q} = -2 \cos \pi g, \tag{181}$$

with $g \in [1, \frac{3}{2}]$ for the $O(N)$ CP, and $g \in [\frac{1}{2}, 1]$ for the low-T $O(N)$, or critical Potts, models; the coupling constant g of the Coulomb gas also yields the central charge:

$$c = 1 - 6(1-g)^2/g. \tag{182}$$

Notice that from the expression (112) of c in terms of $\gamma \leqslant 0$ one arrives at the simple relation:

$$\gamma = 1 - g, \quad g \geqslant 1; \qquad \gamma = 1 - 1/g, \quad g \leqslant 1. \tag{183}$$

The above representation for $N = \sqrt{Q} \in [0, 2]$ gives a range of values $-2 \leqslant c \leqslant 1$; our results also apply for $c \in (-\infty, -2]$, corresponding, e.g., to the $O(N \in [-2, 0])$ branch, with a low-T phase for $g \in [0, \frac{1}{2}]$, and a CP for $g \in [\frac{3}{2}, 2]$.

The fractal dimension $D_{\rm EP}$ of the accessible perimeter, Eq. (165), is, once rewritten in terms of g, and like $c(g)=c(g^{-1})$, a symmetric function of g and g^{-1}

$$D_{\rm EP}=1+\tfrac{1}{2}\,g^{-1}\vartheta(1-g^{-1})+\tfrac{1}{2}\,g\vartheta(1-g), \tag{184}$$

where ϑ is the Heaviside distribution, thus given by two different analytic expressions on either side of the separatrix $g=1$. The dimension of the *hull's frontier*, i.e., the complete set of outer boundary sites of a cluster, has been determined for $O(N)$ and Potts clusters,[63] and reads

$$D_{\rm H}=1+\tfrac{1}{2}\,g^{-1}, \tag{185}$$

for the *entire* range of the coupling constant $g\in[\frac{1}{2},2]$. Comparing to Eq. (184), we therefore see that the accessible perimeter and hull dimensions *coincide* for $g\geqslant 1$, i.e., at the $O(N)$ CP (or for tricritical Potts clusters), whereas they *differ*, namely $D_{\rm EP}<D_H$, for $g<1$, i.e., in the $O(N)$ low-T phase, or for critical Potts clusters. This is the generalization to any Potts model of the effect originally found in percolation.[64] This can be directly understood in terms of the *singly connecting* sites (or bonds) where fjords close in the scaling limit. Their dimension is given by[63]

$$D_{\rm SC}=1+\tfrac{1}{2}\,g^{-1}-\tfrac{3}{2}\,g. \tag{186}$$

Thus, for critical $O(N)$ loops, $g\in(1,2]$ and $D_{\rm SC}<0$, so there exist no closing fjords, which explains the identity:

$$D_{\rm EP}=D_{\rm H}; \tag{187}$$

whereas $D_{\rm SC}>0$, $g\in[\frac{1}{2},1)$ for critical Potts clusters, or in the $O(N)$ low-T phase, where pinching points of positive dimension appear in the scaling limit, so that $D_{\rm EP}<D_{\rm H}$ (Table I). We then find from Eq. (184), with $g\leqslant 1$:

$$(D_{\rm EP}-1)(D_{\rm H}-1)=\tfrac{1}{4}. \tag{188}$$

The symmetry point $D_{\rm EP}=D_{\rm H}=\frac{3}{2}$ corresponds to $g=1$, $N=2$, or $Q=4$, where, as expected, the dimension $D_{\rm SC}=0$ of the pinching points vanishes.

For percolation, described either by $Q=1$, or by the low-T $O(N=1)$ model, with $g=\frac{2}{3}$, we recover the result $D_{\rm EP}=\frac{4}{3}$, recently derived in ref. 38. For the Ising model, described either by $Q=2$, $g=\frac{3}{4}$, or by the $O(N=1)$ CP, $g'=g^{-1}=\frac{4}{3}$, we observe that the EP dimension $D_{\rm EP}=\frac{11}{8}$ coincides, as expected, with that of the critical $O(N=1)$ loops, which in fact appear

Table I. Dimensions for the Critical Q-State Potts Model; $Q=0, 1, 2$ Correspond to Spanning Trees, Percolation and Ising Clusters, Respectively

Q	0	1	2	3	4
c	-2	0	$1/2$	$4/5$	1
D_{EP}	$5/4$	$4/3$	$11/8$	$17/12$	$3/2$
D_{H}	2	$7/4$	$5/3$	$8/5$	$3/2$
D_{SC}	$5/4$	$3/4$	$13/24$	$7/20$	0

as EP's. This is a particular case of a further duality relation between the critical Potts and CP $O(N)$ models:

$$D_{\mathrm{EP}}(Q(g)) = D_{\mathrm{H}}[O(N(g'))], \qquad \text{for} \quad g' = g^{-1}, \qquad g \leqslant 1. \tag{189}$$

In terms of this duality, the central charge takes the simple expression:

$$c = (3-2g)(3-2g'). \tag{190}$$

11. RELATION TO THE SLE_κ PROCESS

An introduction to the stochastic Löwner evolution process (SLE_κ) can be found in ref. 71. This process essentially describes the cluster *hulls* we have introduced above. They can be simple or self-intersecting paths. The SLE_κ is parameterized by κ, which describes the rate of an auxiliary Brownian motion, which is the source to the process. When $\kappa \in [0, 4]$, the random curve is simple, while for $\kappa \in (4, 8)$, the curve is a self-coiling path. For $\kappa \geqslant 8$ the path is space filling. The correspondence to our parameters, the central charge c, the string susceptibility exponent γ, or the Coulomb gas constant g, is as follows.

In the original work by Schramm,[39] the variance of the Gaussian winding angle of a SLE_κ of size R was calculated, and found to be $\sqrt{\kappa \ln R}$. In ref. 73 we found, for instance for the $O(N)$ model, the variance $\sqrt{(4/g) \ln R}$, from which we immediately infer the identity

$$\kappa = \frac{4}{g}. \tag{191}$$

The low-temperature branch $g \in [\frac{1}{2}, 1)$ of the $O(N)$ model, for $N \in [0, 2)$, indeed corresponds to $\kappa \in (4, 8]$ and describes non simple curves, while $N \in [-2, 0]$, $g \in [0, \frac{1}{2}]$ corresponds to $\kappa \geqslant 8$. The critical point branch $g \in [1, \frac{3}{2}]$, $N \in [0, 2]$ gives $\kappa \in [\frac{8}{3}, 4]$, while $g \in [\frac{3}{2}, 2]$, $N \in [-2, 0]$ gives $\kappa \in [2, \frac{8}{3}]$. The range $\kappa \in [0, 2)$ probably corresponds to higher multicritical points with $g > 2$. Owing to Eq. (183) for γ, we have

$$\gamma = 1 - \frac{4}{\kappa}, \quad \kappa \leqslant 4; \qquad \gamma = 1 - \frac{\kappa}{4}, \quad \kappa \geqslant 4. \tag{192}$$

The central charge (112) or (182) is accordingly:

$$c = 1 - 24 \left(\frac{\kappa}{4} - 1 \right)^2 \Big/ \kappa, \tag{193}$$

an expression which of course is symmetric under the *duality* $\kappa/4 \to 4/\kappa = \kappa'$, or

$$\kappa\kappa' = 16,$$

reflecting the symmetry under $gg' = 1$.[34] The self-dual form of the central charge is accordingly:

$$c = \tfrac{1}{4}(6 - \kappa)(6 - \kappa'). \tag{194}$$

From Eqs. (185) and (184) we respectively find

$$D_{\mathrm{H}} = 1 + \frac{1}{8}\kappa, \tag{195}$$

$$D_{\mathrm{EP}} = 1 + \frac{2}{\kappa}\vartheta(\kappa - 4) + \frac{\kappa}{8}\vartheta(4 - \kappa), \tag{196}$$

in agreement with later results in probability theory.[74] For $\kappa \leqslant 4$, we have $D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa)$. For $\kappa \geqslant 4$, the self-coiling scaling paths obey the duality equation (188) derived above, recast here in the context of the SLE_κ process:

$$[D_{\mathrm{EP}}(\kappa) - 1][D_{\mathrm{H}}(\kappa) - 1] = \tfrac{1}{4}, \qquad \kappa \geqslant 4, \tag{197}$$

where now

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa' = 16/\kappa) \qquad \kappa' \leqslant 4.$$

Thus we predict that the external perimeter of a self-coiling $SLE_{\kappa \geqslant 4}$ process is, by *duality*, the simple path of the $SLE_{(16/\kappa) = \kappa' \leqslant 4}$ process.

The symmetric point $\kappa = 4$ corresponds to the $O(N = 2)$ model, or $Q = 4$ Potts model, with $c = 1$. The value $\kappa = 8/3$, $c = 0$ corresponds to a self-avoiding walk, which thus appears[30, 38] as the external frontier of a $\kappa = 6$ process, namely that of a percolation hull.[39, 42]

Work remains to be done to elucidate the relation between the *SLE* construction in probability theory and the Coulomb gas and conformal invariance approaches, as well as the quantum gravity method described here.

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